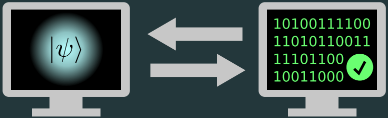


Classical verification of quantum computational advantage

Gregory D. Kahanamoku-Meyer

October 8, 2021

arXiv:1912.05547
arXiv:2104.00687

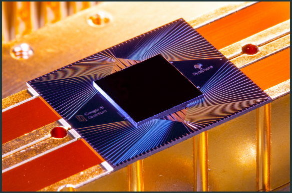
Theory collaborators:

Norman Yao (Berkeley Physics)
Umesh Vazirani (Berkeley CS)
Soonwon Choi (MIT Physics)

Berkeley
UNIVERSITY OF CALIFORNIA

Quantum computational advantage

Recent experimental demonstrations:



Random circuit sampling
[Arute et al., Nature '19]

Gaussian boson sampling
[Zhong et al., Science '20]

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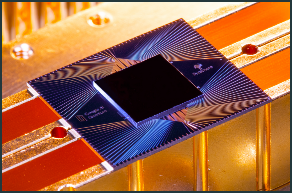

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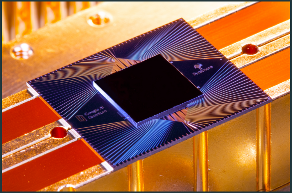

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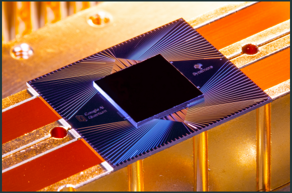

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Largest experiments → impossible to classically simulate

> "… [Rule] out alternative [classical] hypotheses that might be plausible in this experiment" [Zhong et al.]
> Quantum is the only reasonable explanation for observed behavior

“Black-box” proofs of quantumness

Stronger: rule out all classical hypotheses, even adversarial!

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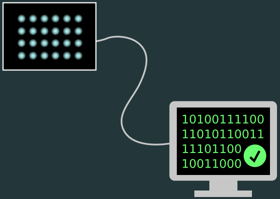

Local: powerfully refute the extended Church-Turing thesis

“Black-box” proofs of quantumness

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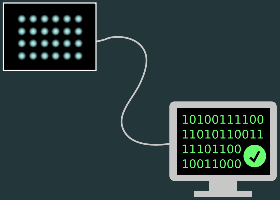

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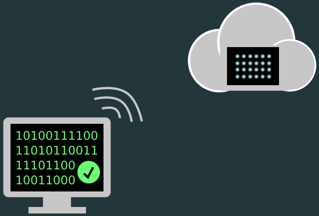

Remote: validate an untrusted quantum cloud service

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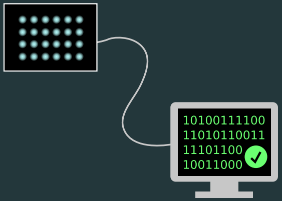

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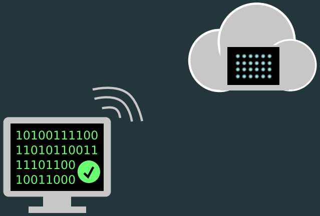

Remote: validate an untrusted quantum cloud service

Proof not specific to quantum mechanics: disprove null hypothesis that output was generated classically.

Need computational assumption—really an “argument”

“Black-box” proofs of quantumness

Efficiently-verifiable test that only quantum computers can pass.

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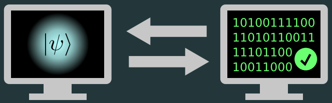
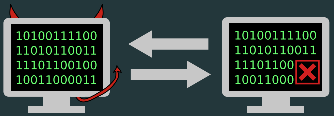
For polynomially-bounded classical verifier:



Completeness

$\exists$ BQP prover s.t. Verifier accepts w.p. $> 2/3$



Soundness

$\forall$ BPP provers, Verifier accepts w.p. $< 1/3$

NISQ verifiable quantum advantage

Trivial solution: integer factorization

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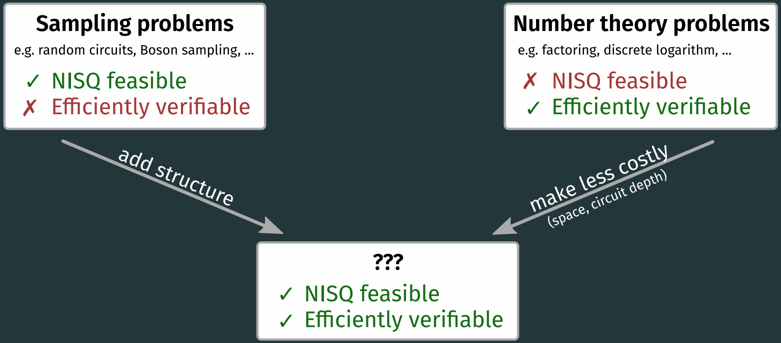
NISQ: Noisy Intermediate-Scale Quantum devices



Sampling problems

e.g. random circuits, Boson sampling, ...

✓ NISQ feasible

✗ Efficiently verifiable

Number theory problems

e.g. factoring, discrete logarithm, ...

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add structure

make less costly
(space, circuit depth)

???

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Sampling problems

Task: generate samples from a “hard” probability distribution.

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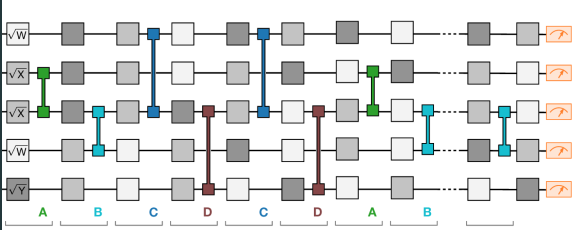

Arute et al. 2019

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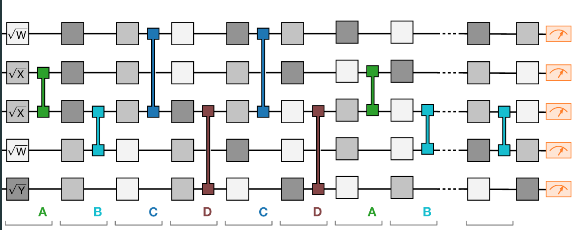

Arute et al. 2019

- Specify distribution via a quantum circuit
- Intuitive classical hardness: no structure → need to simulate quantum, which is hard

Adding structure to sampling problems

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IQP circuits [Shepherd and Bremner, '08]:

- Hide a secret string $\mathbf{s}$ in the quantum circuit
- Set up circuit so it is *biased* to generate samples $\mathbf{x}$ with $\mathbf{x}^\intercal \cdot \mathbf{s} = 0$.

IQP circuits [Shepherd and Bremner, ’08]

Consider a matrix $P \in \{0,1\}^{k\times n}$ and “action” θ.

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Example:

$$H = X_0 X_1 X_3 + X_1 X_2 X_4 X_5 + \cdots \tag{1}$$

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Bremner, Jozsa, Shepherd ’11: classically sampling IQP circuits would collapse polynomial heirarchy

Bremner, Montanaro, Shepherd ’16: average case is likely hard as well

IQP proof of quantumness [Shepherd and Bremner, ’08]

Let $\theta = \pi/8$ and P have the form:

$$P = \left[\begin{array}{c} G \\ \hline R \end{array}\right]$$

$G^{\intercal}$ is generator of Quadratic Residue code, R random.

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Conjecture [SB ’08]: Scrambling P cryptographically hides G (and equivalently s)

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Assuming s hidden, can classical do better than 0.5? **Try to take advantage properties of embedded code.**

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QR code codewords are 50% even parity, 50% odd parity.

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$$y \cdot s = (Gd) \cdot (Ge) \pmod 2$$

Fact: $(Gd) \cdot (Ge) = 1$ iff Gd, Ge both have odd parity.

IQP: Classical strategy [SB ’08]

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Classical: $\Pr[Y^\intercal \cdot s = 0] = 0.75$

Consider choosing random $\boldsymbol{d}, \boldsymbol{e} \xleftarrow{\$} \{0,1\}^n$, and letting

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Then:

$$\boldsymbol{y} \cdot \boldsymbol{s} = (G\boldsymbol{d}) \cdot (G\boldsymbol{e}) \pmod 2$$

Fact: $(G\boldsymbol{d}) \cdot (G\boldsymbol{e}) = 1$ iff $G\boldsymbol{d}$, $G\boldsymbol{e}$ both have odd parity.

Thus $\boldsymbol{y} \cdot \boldsymbol{s} = 0$ with probability 3/4!

IQP: Can we do better classically? [GDKM ’19 arXiv:1912.05547]

Key: Correlate samples to attack the key s

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Let y_i form rows of a matrix M, such that $M\boldsymbol{s} = \mathbf{0}$

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Can solve for $\boldsymbol{s}$! ... If M has high rank. Empirically it does!

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- Could pick a different G for which this attack would not succeed?
- Ultimately, would like to rely on standard cryptographic assumptions...

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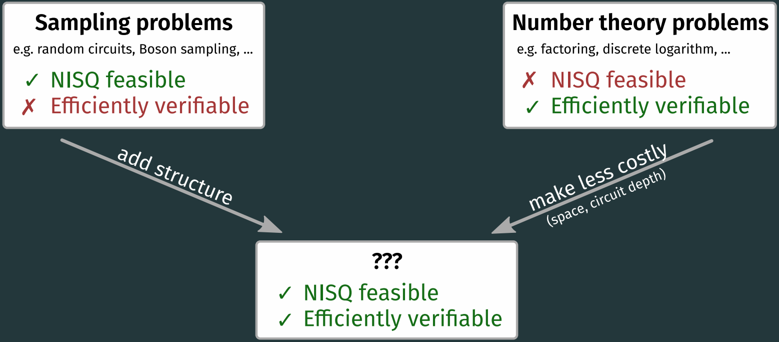

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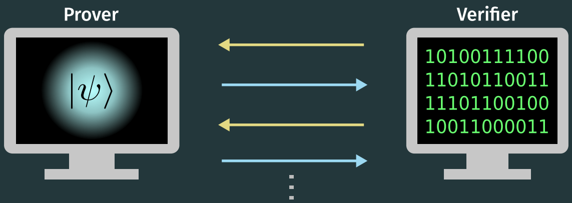
Interactive proofs of quantumness

Multiple rounds of interaction between the prover and verifier

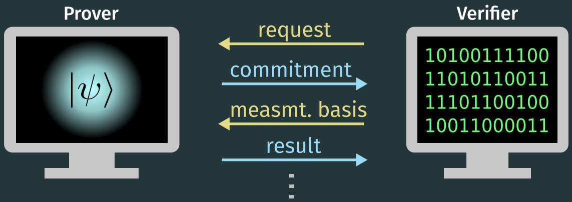


Round 1: Prover commits to a specific quantum state

Round 2+: Verifier asks for measurement in specific basis

Interactive proofs of quantumness

Multiple rounds of interaction between the prover and verifier



Round 1: Prover commits to a specific quantum state

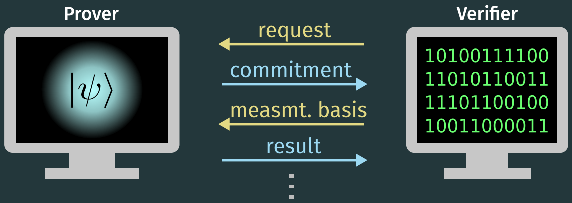
Round 2+: Verifier asks for measurement in specific basis

By randomizing choice of basis and repeating interaction, can ensure prover would respond correctly in *any* basis

Brakerski, Christiano, Mahadev, Vidick, Vazirani '18 (arXiv:1804.00640).

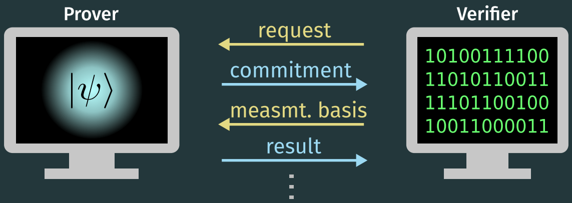
Can be extended to verify arbitrary quantum computations! (arXiv:1804.01082)

Interactive proofs of quantumness



From a proof of security perspective:

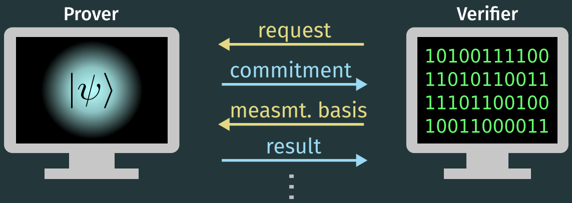
Interactive proofs of quantumness



From a proof of security perspective:

- **Classical** cheater can be rewound: extract measurement results in all choices of basis

Interactive proofs of quantumness



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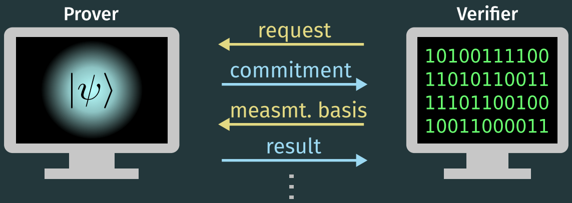

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"Rewinding" proof of hardness doesn't go through for quantum prover—can use post-quantum cryptography!

State commitment (round 1): trapdoor claw-free functions

How does the prover commit to a state?

Consider a trapdoor claw-free function family (TCF) (Gen, $\{(f_i, T_i)\}$).

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Prover		**Verifier**
Evaluate f_i on uniform superposition $\sum_x \lvert x\rangle \lvert f_i(x)\rangle$	$\xleftarrow{\quad f_i \quad}$	$(f_i, t_i) \leftarrow \mathrm{Gen}(1^\lambda)$
Measure 2^{nd} register as y	$\xrightarrow{\quad y \quad}$	Store y as commitment compute $(x_0, x_1) \leftarrow T_i(y, t_i)$

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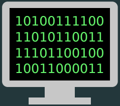

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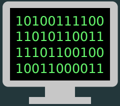
Prover has committed to the state $(|x_0\rangle + |x_1\rangle) |y\rangle$

BCMVV ’18 protocol

Prover		Verifier
		
Evaluate f on uniform superposition: $\sum_x \|x\rangle \|f(x)\rangle$	$\xleftarrow{\quad f \quad}$	Pick trapdoor claw-free function f
Measure 2nd register as y	$\xrightarrow{\quad y \quad}$	Compute x_0, x_1 from y using trapdoor

Brakerski, Christiano, Mahadev, Vidick, Vazirani ’18 (arXiv:1804.00640)

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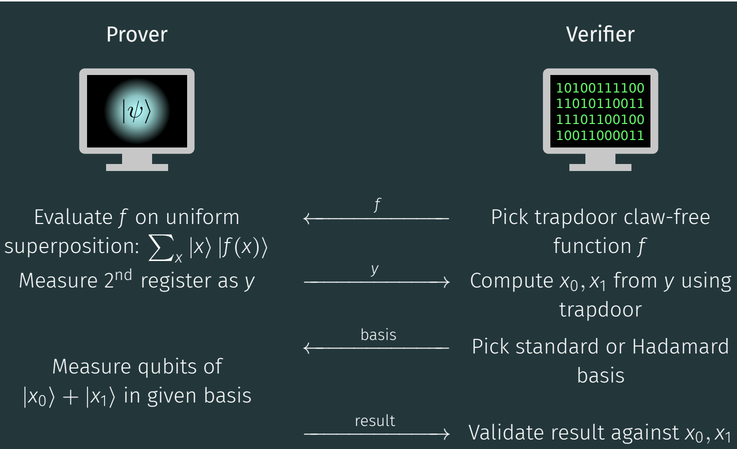

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$\xleftarrow{\quad \text{basis} \quad}$ Pick standard or Hadamard basis

Measure qubits of $|x_0\rangle + |x_1\rangle$ in given basis

$\xrightarrow{\quad \text{result} \quad}$ Validate result against x_0, x_1

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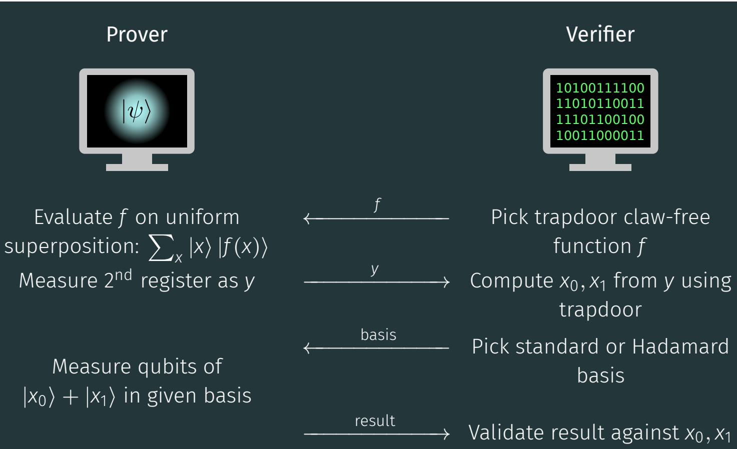

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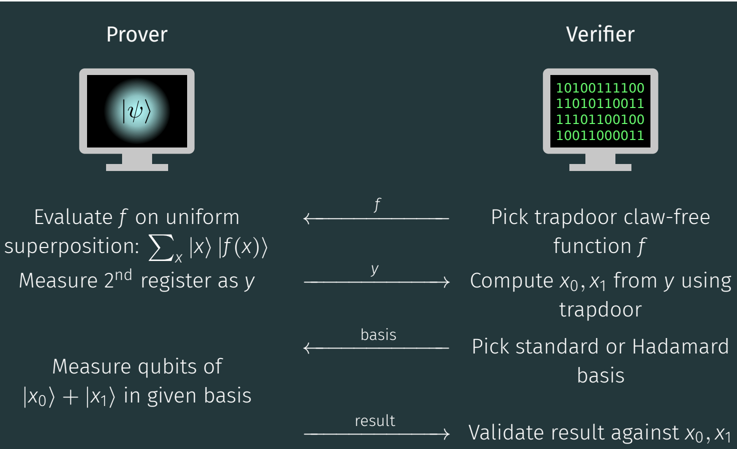

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Learning-with-Errors TCF has adaptive hardcore bit

Brakerski, Christiano, Mahadev, Vidick, Vazirani ’18 (arXiv:1804.00640)

BCMVV ’18 protocol: adaptive hardcore bit

State after commitment round: $(|x_0\rangle + |x_1\rangle)\,|y\rangle$

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Adaptive hardcore bit:

Computationally hard to generate a tuple (y, x_0, d, b) such that:

$$d \cdot (x_0 + x_1) = b$$

$$f_i(x_0) = f_i(x_1) = y$$

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Note: AHCB can be post-quantum secure and protocol still works!

Trapdoor claw-free functions

TCF	Trapdoor	Claw-free	Adaptive hard-core bit
LWE [1]	✓	✓	✓
Ring-LWE [2]	✓	✓	✗
$x^2 \bmod N$ [3]	✓	✓	✗
DDH [3]	✓	✓	✗

[1] Brakerski, Christiano, Mahadev, Vazirani, Vidick ’18 (arXiv:1804.00640)

[2] Brakerski, Koppula, Vazirani, Vidick ’20 (arXiv:2005.04826)

[3] GDKM, Choi, Vazirani, Yao ’21 (arXiv:2104.00687)

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BKVV '20 [2]: Non-interactive protocol without adaptive hardcore bit, in random oracle model

$$d \cdot (x_0 \oplus x_1) = H(x_0) \oplus H(x_1)$$

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Can we remove AHCB in the standard model?

[1] Brakerski, Christiano, Mahadev, Vazirani, Vidick '18 (arXiv:1804.00640)

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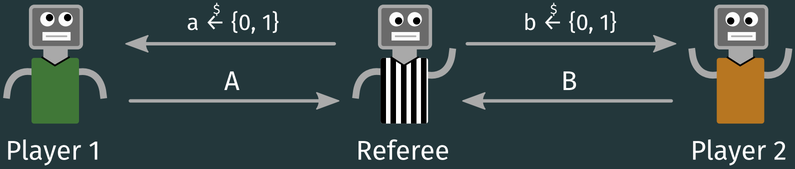
The CHSH game (Bell test)

Two-player cooperative game.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$b \overset{\$}{\leftarrow} \{0, 1\}$
A
B
Player 1
Referee
Player 2

Players win if $A \oplus B = a \cdot b$

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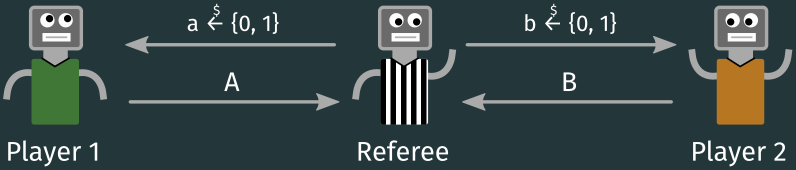
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Classical optimal strategy: return equal values, hope $a \cdot b = 0$.
75% success rate.

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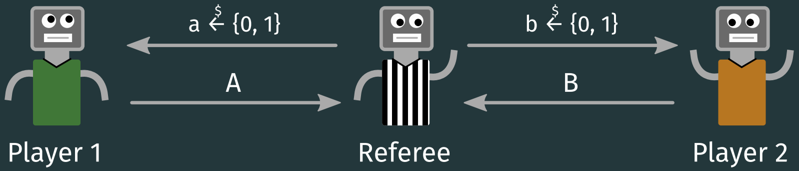
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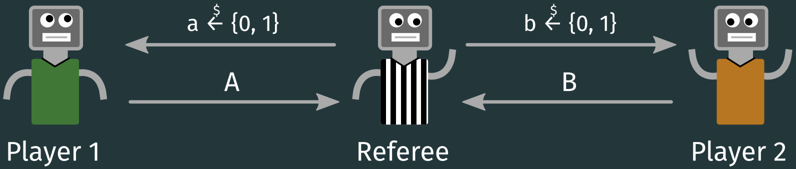

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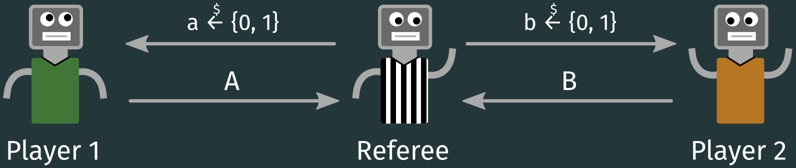

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Aligned basis $\rightarrow$ same result; antialigned $\rightarrow$ opposite result!

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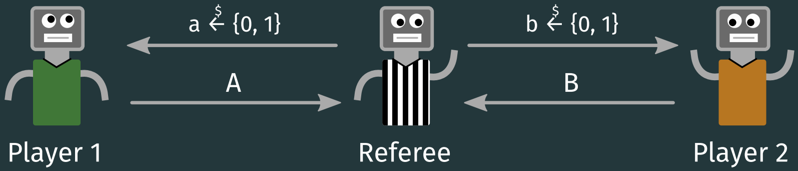

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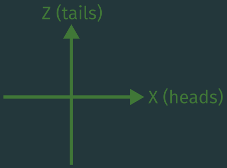

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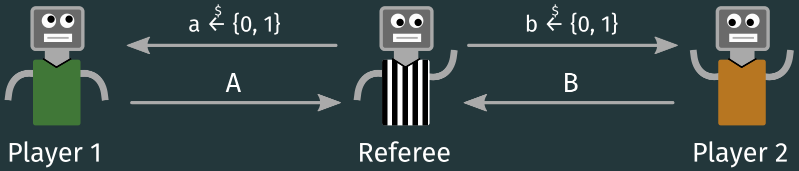

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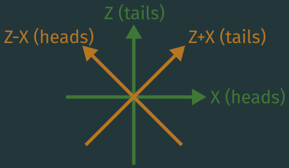

Z (tails)

Z-X (heads)

Z+X (tails)

X (heads)

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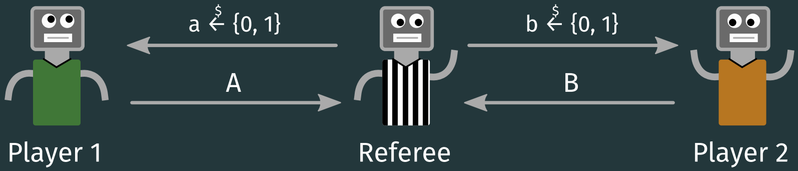

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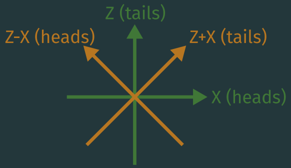
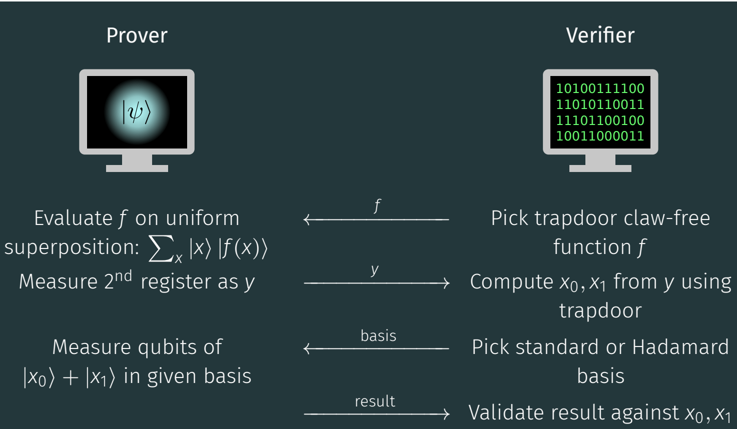


Quantum: $\cos^2(\pi/8) \approx 85\%$
Classical: 75%

BCMVV ’18 protocol



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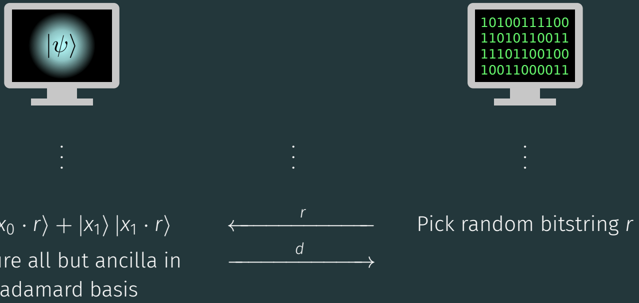
Replace Hadamard basis measurement with “1-player CHSH”

Brakerski, Christiano, Mahadev, Vidick, Vazirani ’18 (arXiv:1804.00640)

Interactive measurement: computational Bell test

Replace Hadamard basis measurement with two-step process: "condense" x_0, x_1 into a single qubit, and then do a "Bell test."



$|x_0\rangle |x_0 \cdot r\rangle + |x_1\rangle |x_1 \cdot r\rangle$ $\xleftarrow{r}$ Pick random bitstring r

Measure all but ancilla in Hadamard basis $\xrightarrow{d}$

GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

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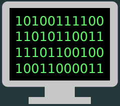

⋮ ⋮ ⋮

$|x_0\rangle\,|x_0 \cdot r\rangle + |x_1\rangle\,|x_1 \cdot r\rangle$ $\xleftarrow{\quad r \quad}$ Pick random bitstring r

Measure all but ancilla in Hadamard basis $\xrightarrow{\quad d \quad}$

Single-qubit state: $|x_0 \cdot r\rangle + |x_1 \cdot r\rangle$

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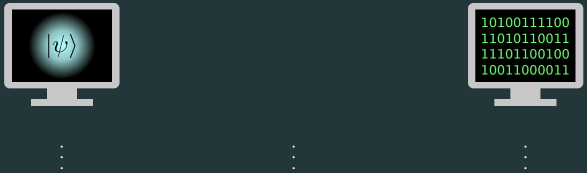

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Single-qubit state: $|\uparrow\rangle$ or $|\downarrow\rangle$ if $x_0 \cdot r = x_1 \cdot r$, otherwise $|\leftarrow\rangle$ or $|\rightarrow\rangle$.

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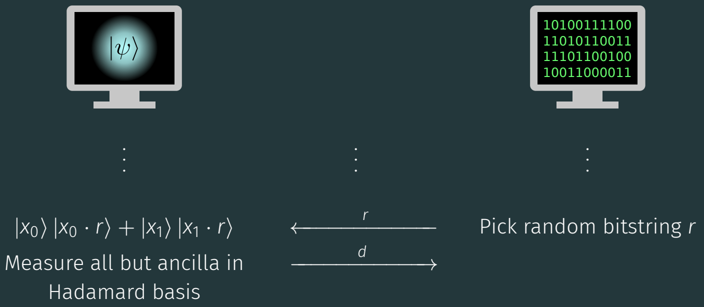

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Single-qubit state: $|{\uparrow}\rangle$ or $|{\downarrow}\rangle$ if $x_0 \cdot r = x_1 \cdot r$, otherwise $|{\leftarrow}\rangle$ or $|{\rightarrow}\rangle$.
Polarization hidden via:

Cryptographic secret (here) $\Leftrightarrow$ Non-communication (Bell test)

GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

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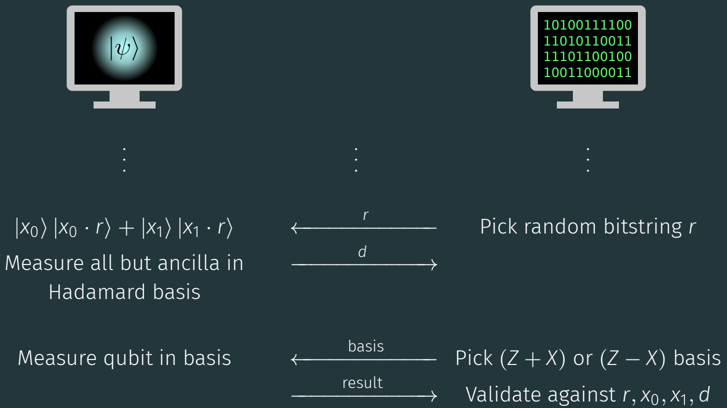

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$|x_0\rangle\,|x_0 \cdot r\rangle + |x_1\rangle\,|x_1 \cdot r\rangle$ $\xleftarrow{\quad r \quad}$ Pick random bitstring r

Measure all but ancilla in Hadamard basis $\xrightarrow{\quad d \quad}$

Measure qubit in basis $\xleftarrow{\text{basis}}$ Pick $(Z+X)$ or $(Z-X)$ basis

$\xrightarrow{\text{result}}$ Validate against r, x_0, x_1, d

GDKM, Choi, Vazirani, Yao ’21 (arXiv:2104.00687)

Computational Bell test: classical bound

Run protocol many times, collect statistics.

p_s: Success rate for standard basis measurement.

p_{CHSH}: Success rate when performing CHSH-type measurement.

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Soundness (classical bound): $p_s + 4p_{\text{CHSH}} - 4 < \text{negl}(n)$

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$p_s + 4p_{\text{CHSH}} - 4 = \sqrt{2} - 1 \approx$ **0.414**

Computational Bell test: classical bound

Run protocol many times, collect statistics.

p_s: Success rate for standard basis measurement.

p_{CHSH}: Success rate when performing CHSH-type measurement.

Under assumption of claw-free function:

Soundness (classical bound): $p_s + 4p_{\mathrm{CHSH}} - 4 < \mathrm{negl}(n)$

Completeness (ideal quantum): $p_s = 1, p_{\mathrm{CHSH}} = \cos^2(\pi/8)$

$p_s + 4p_{\mathrm{CHSH}} - 4 = \sqrt{2} - 1 \approx$ **0.414**

Note: Let $p_s = 1$. Then for p_{CHSH}:

Classical bound 75%, ideal quantum $\sim$ 85%. Same as regular CHSH!

GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

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See arXiv:2104.00687 for details

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  - Measurement scheme removes need for *reversibility* in quantum circuits—significant efficiency gains

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TCF constructions

TCF	A.H.C.B.	Gate count	n for hardness
LWE [1]	✓	$\mathcal{O}(n^2 \log^2 n)$	10^4
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A.H.C.B. = ”adaptive hard core bit”

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Q. advantage in 10^6 Toffoli gates

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Security proof: Given $g^{\mathbf{M}}$, DDH hides rank of $\mathbf{M}$. Inversion would imply algorithm to determine if $\mathbf{M}$ is full rank.

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- Reversible Euclidean algorithm is hard, maybe irreversible optimization can help?

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"In the box" ideas (not necessarily bad):

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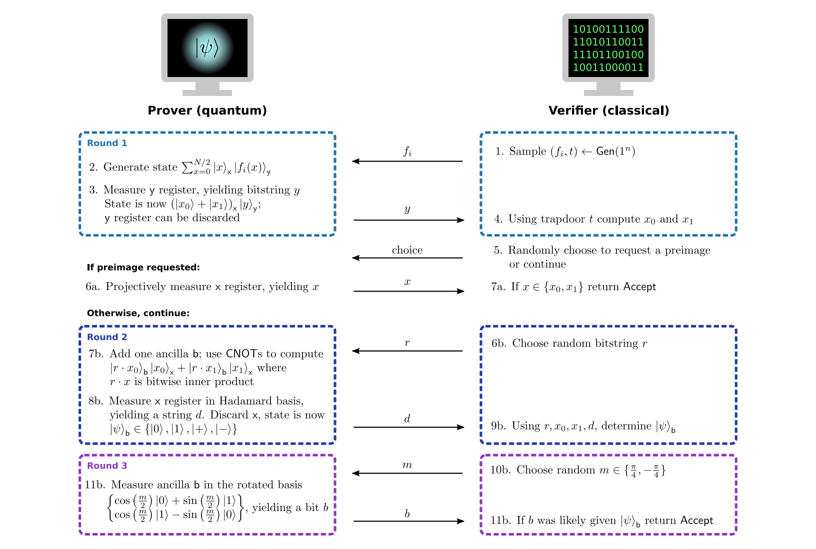
Way outside the box?

Backup!

Full protocol



Prover (quantum)

Verifier (classical)

Round 1

1. Sample $(f_i, t) \leftarrow \mathsf{Gen}(1^n)$

$\xleftarrow{f_i}$

2. Generate state $\sum_{x=0}^{N/2} |x\rangle_{\mathsf{x}} |f_i(x)\rangle_{\mathsf{y}}$

3. Measure y register, yielding bitstring y
State is now $(|x_0\rangle + |x_1\rangle)_{\mathsf{x}} |y\rangle_{\mathsf{y}}$;
y register can be discarded

$\xrightarrow{y}$

4. Using trapdoor t compute x_0 and x_1

5. Randomly choose to request a preimage or continue

$\xleftarrow{\text{choice}}$

If preimage requested:

6a. Projectively measure x register, yielding x

$\xrightarrow{x}$

7a. If $x \in \{x_0, x_1\}$ return Accept

Otherwise, continue:

Round 2

6b. Choose random bitstring r

$\xleftarrow{r}$

7b. Add one ancilla b; use CNOTs to compute
$|r \cdot x_0\rangle_{\mathsf{b}} |x_0\rangle_{\mathsf{x}} + |r \cdot x_1\rangle_{\mathsf{b}} |x_1\rangle_{\mathsf{x}}$ where
$r \cdot x$ is bitwise inner product

8b. Measure x register in Hadamard basis, yielding a string d. Discard x, state is now
$|\psi\rangle_{\mathsf{b}} \in \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$

$\xrightarrow{d}$

9b. Using r, x_0, x_1, d, determine $|\psi\rangle_{\mathsf{b}}$

Round 3

10b. Choose random $m \in \{\frac{\pi}{4}, -\frac{\pi}{4}\}$

$\xleftarrow{m}$

11b. Measure ancilla b in the rotated basis
$\left\{ \begin{matrix} \cos\left(\frac{m}{2}\right)|0\rangle + \sin\left(\frac{m}{2}\right)|1\rangle \\ \cos\left(\frac{m}{2}\right)|1\rangle - \sin\left(\frac{m}{2}\right)|0\rangle \end{matrix} \right\}$, yielding a bit b

$\xrightarrow{b}$

11b. If b was likely given $|\psi\rangle_{\mathsf{b}}$ return Accept

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When we generate $\sum_x |x\rangle\,|f(x)\rangle$, **add redundancy to $f(x)$, for bit flip error detection!**

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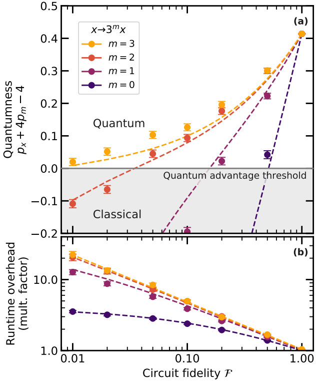

Numerical results for $x^2 \bmod N$ with $\log N = 512$ bits.

Here: make transformation $x^2 \bmod N \Rightarrow (kx)^2 \bmod k^2 N$

Partial measurements in the lab

TIQI

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Prof. Christopher Monroe

Dr. Daiwei Zhu

Dr. Crystal Noel

and others!

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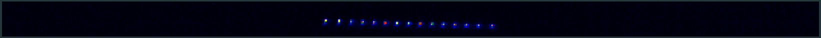

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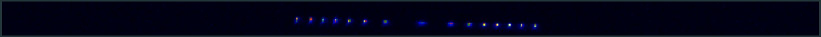

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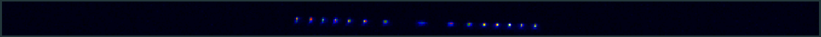

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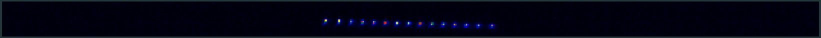

Improving circuit sizes

Most demanding step in all these protocols: evaluating TCF

$$\mathcal{U}_f \left|x\right\rangle \left|0^{\otimes n}\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$$

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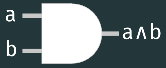
Protocol allows us to make circuits irreversible!

Technique: taking out the garbage

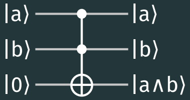
Goal: $\mathcal{U}_f \left|x\right\rangle \left|0^{\otimes n}\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$

When converting classical circuits to quantum:

Garbage bits: extra entangled outputs due to unitarity



Classical AND



Quantum AND (Toffoli)

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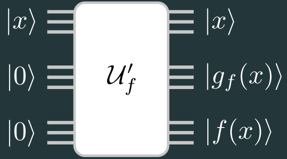

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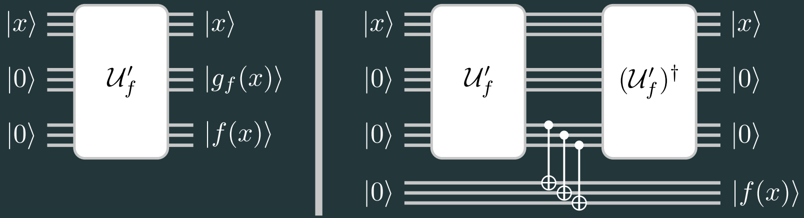

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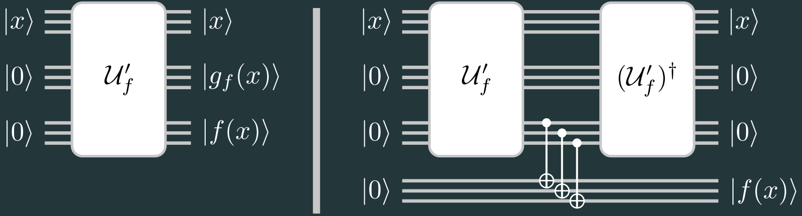

Lots of time and space overhead!

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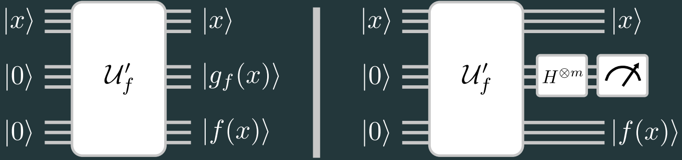

Can we “measure them away” instead?

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Measure garbage bits $g_f(x)$ in Hadamard basis, get some string h. End up with state:

$$\sum_x (-1)^{h \cdot g_f(x)} |x\rangle |f(x)\rangle$$

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1024-bit $x^2 \bmod N$ costs only 10^6 Toffoli gates.