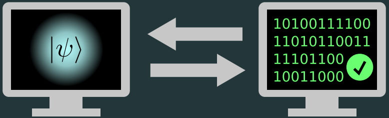


Classical verification of quantum computational advantage

Gregory D. Kahanamoku-Meyer

November 10, 2021

arXiv:1912.05547
arXiv:2104.00687

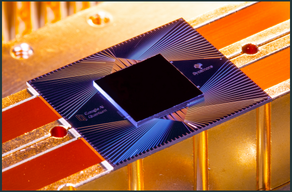
Theory collaborators:

Norman Yao (Berkeley Physics)
Umesh Vazirani (Berkeley CS)
Soonwon Choi (MIT Physics)

Berkeley
UNIVERSITY OF CALIFORNIA

Quantum computational advantage

Recent experimental demonstrations:

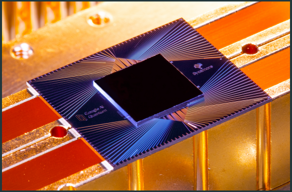


Random circuit sampling
[Arute et al., Nature '19]

Gaussian boson sampling
[Zhong et al., Science '20]

Quantum computational advantage

Recent experimental demonstrations:



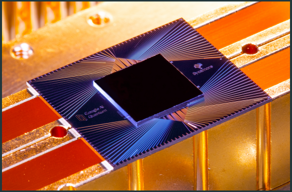
Random circuit sampling
[Arute et al., Nature '19]

Gaussian boson sampling
[Zhong et al., Science '20]

Largest experiments → “impossible” to classically simulate

Quantum computational advantage

Recent experimental demonstrations:



Random circuit sampling
[Arute et al., Nature '19]

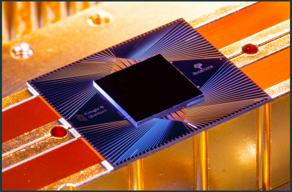
Gaussian boson sampling
[Zhong et al., Science '20]

Largest experiments $\rightarrow$ "impossible" to classically simulate

> "... [Rule] out alternative [classical] hypotheses that might be plausible in this experiment" [Zhong et al.]

Quantum computational advantage

Recent experimental demonstrations:



Random circuit sampling
[Arute et al., Nature '19]

Gaussian boson sampling
[Zhong et al., Science '20]

Largest experiments $\rightarrow$ “impossible” to classically simulate

“... [Rule] out alternative [classical] hypotheses that might be plausible in this experiment” [Zhong et al.]

Quantum is the only reasonable explanation for observed behavior

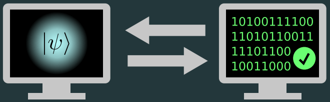
“Black-box” proofs of quantumness

Efficiently-verifiable test that only quantum computers can pass.

“Black-box” proofs of quantumness

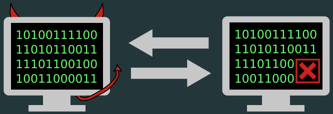
Efficiently-verifiable test that only quantum computers can pass.

For polynomially-bounded classical verifier:



Completeness

$\exists$ BQP prover s.t. Verifier accepts w.p. $> 2/3$



Soundness

$\forall$ BPP provers, Verifier accepts w.p. $< 1/3$

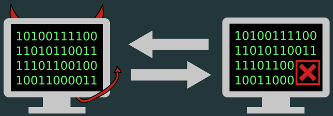
“Black-box” proofs of quantumness

Efficiently-verifiable test that only quantum computers can pass.

For polynomially-bounded classical verifier:

Completeness

$\exists$ BQP prover s.t. Verifier accepts w.p. $> 2/3$



Soundness

$\forall$ BPP provers, Verifier accepts w.p. $< 1/3$

Fully classical verifier (and comms.),

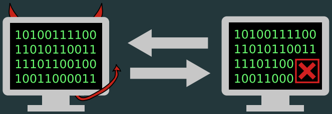
"Black-box" proofs of quantumness

Efficiently-verifiable test that only quantum computers can pass.

For polynomially-bounded classical verifier:

Completeness

$\exists$ BQP prover s.t. Verifier accepts w.p. $> 2/3$



Soundness

$\forall$ BPP provers, Verifier accepts w.p. $< 1/3$

Fully classical verifier (and comms.), single black-box prover,

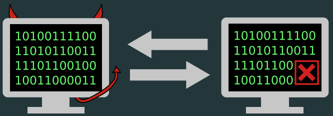
"Black-box" proofs of quantumness

Efficiently-verifiable test that only quantum computers can pass.

For polynomially-bounded classical verifier:

Completeness

$\exists$ BQP prover s.t. Verifier accepts w.p. $> 2/3$



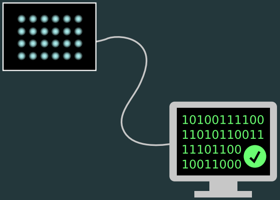
Soundness

$\forall$ BPP provers, Verifier accepts w.p. $< 1/3$

Fully classical verifier (and comms.), single black-box prover, superpolynomial computational separation

“Black-box” proofs of quantumness

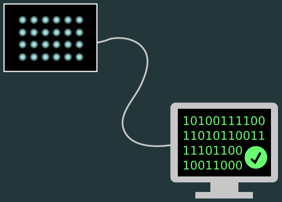
Efficiently-verifiable test that only quantum computers can pass.



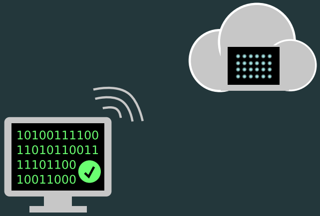
Local: powerfully refute the
extended Church-Turing thesis

“Black-box” proofs of quantumness

Efficiently-verifiable test that only quantum computers can pass.



Local: powerfully refute the extended Church-Turing thesis



Remote: validate an untrusted quantum cloud service

NISQ verifiable quantum advantage

Trivial solution: Shor's algorithm

NISQ verifiable quantum advantage

Trivial solution: Shor's algorithm... but we want to do near-term!

NISQ verifiable quantum advantage

Trivial solution: Shor's algorithm... but we want to do near-term!

Sampling problems

e.g. random circuits, Boson sampling, ...

✓ NISQ feasible
✗ Efficiently verifiable

Number theory problems

e.g. factoring, discrete logarithm, ...

✗ NISQ feasible
✓ Efficiently verifiable

add structure

make less costly

???

✓ NISQ feasible
✓ Efficiently verifiable

Adding structure to sampling problems

Idea: some *property* of samples that we can check?

Adding structure to sampling problems

Idea: some *property* of samples that we can check?

Generically: seems difficult to make work.

The point of random circuits is that they don't have structure!

Adding structure to sampling problems

Idea: some *property* of samples that we can check?

Generically: seems difficult to make work.

> The point of random circuits is that they don't have structure!

IQP circuits [Shepherd and Bremner, '08]:

- Hide a secret string $\mathbf{s}$ in the quantum circuit
- Set up circuit so it is *biased* to generate samples $\mathbf{x}$ with $\mathbf{x}^\intercal \cdot \mathbf{s} = 0$.

IQP circuits [Shepherd and Bremner, '08]

Consider a matrix $P \in \{0,1\}^{k\times n}$ and "action" θ.

IQP circuits [Shepherd and Bremner, '08]

Consider a matrix $P \in \{0,1\}^{k \times n}$ and "action" θ.

Let $H = \sum_i \prod_j X_j^{P_{ij}}$.

Example:

$$H = X_0 X_1 X_3 + X_1 X_2 X_4 X_5 + \cdots \tag{1}$$

IQP circuits [Shepherd and Bremner, '08]

Consider a matrix $P \in \{0,1\}^{k \times n}$ and "action" θ.

Let $H = \sum_i \prod_j X_j^{P_{ij}}$.

Example:

$$H = X_0 X_1 X_3 + X_1 X_2 X_4 X_5 + \cdots \tag{1}$$

Distribution of sampling result X:

$$\Pr[X = x] = \left| \left\langle x \,\middle|\, e^{-iH\theta} \,\middle|\, 0 \right\rangle \right|^2 \tag{2}$$

IQP circuits [Shepherd and Bremner, ’08]

Consider a matrix $P \in \{0,1\}^{k \times n}$ and “action” θ.

Let $H = \sum_i \prod_j X_j^{P_{ij}}$.

Example:

$$H = X_0 X_1 X_3 + X_1 X_2 X_4 X_5 + \cdots \tag{1}$$

Distribution of sampling result $\boldsymbol{X}$:

$$\Pr[\boldsymbol{X} = \boldsymbol{x}] = \left| \left\langle \boldsymbol{x} \,\middle|\, e^{-iH\theta} \,\middle|\, \boldsymbol{0} \right\rangle \right|^2 \tag{2}$$

Bremner, Jozsa, Shepherd ’11: classically sampling worst-case IQP circuits would collapse polynomial heirarchy

Bremner, Montanaro, Shepherd ’16: average case is likely hard as well

IQP proof of quantumness [Shepherd and Bremner, ’08]

Let $\theta = \pi/8$, and s (secret) and P have the form:

$$P = \left[\begin{array}{c} G \\ \hline R \end{array}\right]$$

$G^\intercal$ is generator of Quadratic Residue code, R random.

IQP proof of quantumness [Shepherd and Bremner, '08]

Let $\theta = \pi/8$, and s (secret) and P have the form:

$$P = \begin{bmatrix} G \\ \hline R \end{bmatrix} \qquad P\boldsymbol{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$G^\intercal$ is generator of Quadratic Residue code, R random.

$$\Pr[X^\intercal \cdot \boldsymbol{s} = 0] = \mathop{\mathbb{E}}_{x}\left[\cos^2\left(\frac{\pi}{8}(1 - 2\mathrm{wt}(G\boldsymbol{x}))\right)\right]$$

IQP proof of quantumness [Shepherd and Bremner, '08]

Let $\theta = \pi/8$, and s (secret) and P have the form:

$$P = \begin{bmatrix} G \\ \hline R \end{bmatrix} \qquad P\boldsymbol{s} = \begin{bmatrix} 1\\1\\1\\1\\1\\1\\0\\0\\0\\0\\0\\0 \end{bmatrix}$$

$G^{\intercal}$ is generator of Quadratic Residue code, R random.

$$\Pr[X^{\intercal} \cdot \boldsymbol{s} = 0] = \mathop{\mathbb{E}}_{x}\left[\cos^2\left(\frac{\pi}{8}(1 - 2\mathrm{wt}(G\boldsymbol{x}))\right)\right]$$

QR code: codewords have $\mathrm{wt}(\boldsymbol{c}) \bmod 4 \in \{0, -1\}$

IQP proof of quantumness [Shepherd and Bremner, ’08]

Let $\theta = \pi/8$, and s (secret) and P have the form:

$$P = \left[\begin{array}{c} G \\ \hline R \end{array}\right] \qquad P\mathbf{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$G^\intercal$ is generator of Quadratic Residue code, R random.

$$\Pr[X^\intercal \cdot \mathbf{s} = 0] = \cos^2\left(\frac{\pi}{8}\right) \approx 0.85$$

QR code: codewords have $\mathrm{wt}(\mathbf{c}) \bmod 4 \in \{0, -1\}$

IQP: Hiding s

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Best classical: $\Pr[Y^\intercal \cdot s = 0] = ?$

$$P = \begin{bmatrix} G \\ \hline R \end{bmatrix} \qquad P\boldsymbol{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

IQP: Hiding s

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Best classical: $\Pr[Y^\intercal \cdot s = 0] = ?$

$$P = \begin{bmatrix} G \\ \hline R \end{bmatrix} \qquad Ps = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\substack{\text{permute rows,} \\ \text{Gauss-Jordan} \\ \text{columns}}} P's' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Scrambling preserves quantum success rate.

IQP: Hiding s

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Best classical: $\Pr[Y^\intercal \cdot s = 0] = ?$

$$P = \left[\begin{array}{c} G \\ \hline R \end{array}\right] \qquad P\mathbf{s} = \begin{bmatrix}1\\1\\1\\1\\1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix} \xrightarrow{\text{permute rows, Gauss-Jordan columns}} P'\mathbf{s'} = \begin{bmatrix}1\\0\\0\\1\\0\\1\\0\\1\\0\\0\\1\\1\end{bmatrix}$$

Scrambling preserves quantum success rate.

Conjecture [SB ’08]: Scrambling P cryptographically hides G (and equivalently s)

IQP: Classical strategy

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Best classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Assuming s hidden, can classical do better than 0.5? **Try to take advantage properties of embedded code.**

IQP: Classical strategy

$$\text{Quantum: } \Pr[X^\intercal \cdot s = 0] \approx 0.85$$
$$\text{Best classical: } \Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$$

Assuming s hidden, can classical do better than 0.5? **Try to take advantage properties of embedded code.**

Consider choosing random $d \stackrel{\$}{\leftarrow} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = 1}} p$$

IQP: Classical strategy

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Best classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Assuming s hidden, can classical do better than 0.5? **Try to take advantage properties of embedded code.**

Consider choosing random $d \xleftarrow{\$} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = 1}} p$$

Then:

$$y \cdot s = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = 1}} p \cdot s \pmod 2$$

IQP: Classical strategy

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Best classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Assuming s hidden, can classical do better than 0.5? **Try to take advantage properties of embedded code.**

Consider choosing random $d \xleftarrow{\$} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = 1}} p$$

Then:

$$y \cdot s = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = p \cdot s = 1}} 1 \pmod 2$$

IQP: Classical strategy

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Best classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Assuming s hidden, can classical do better than 0.5? **Try to take advantage properties of embedded code.**

Consider choosing random $d \xleftarrow{\$} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = 1}} p$$

Then:

$$y \cdot s = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot s = 1}} p \cdot d \pmod{2}$$

IQP: Classical strategy

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Best classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Assuming s hidden, can classical do better than 0.5? **Try to take advantage properties of embedded code.**

Consider choosing random $d \xleftarrow{\$} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = 1}} p$$

Then:

$$y \cdot s = \mathrm{wt}(Gd) \pmod 2$$

QR code codewords are 50% even parity, 50% odd parity.

IQP: Classical strategy [SB '08]

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Consider choosing random $d, e \stackrel{\$}{\leftarrow} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = p \cdot e = 1}} p$$

IQP: Classical strategy [SB '08]

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Consider choosing random $d, e \xleftarrow{\$} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = p \cdot e = 1}} p$$

Then:

IQP: Classical strategy [SB '08]

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Consider choosing random $d, e \xleftarrow{\$} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = p \cdot e = 1}} p$$

Then:

$$y \cdot s = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = p \cdot e = 1}} p \cdot s \pmod 2$$

IQP: Classical strategy [SB '08]

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Consider choosing random $d, e \xleftarrow{\$} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = p \cdot e = 1}} p$$

Then:

$$y \cdot s = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot s = 1}} (p \cdot d)(p \cdot e) \pmod 2$$

IQP: Classical strategy [SB '08]

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Classical: $\Pr[Y^\intercal \cdot s = 0] \stackrel{?}{=} 0.5$

Consider choosing random $d, e \xleftarrow{\$} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = p \cdot e = 1}} p$$

Then:

$$y \cdot s = (Gd) \cdot (Ge) \pmod 2$$

Fact: $(Gd) \cdot (Ge) = 1$ iff Gd, Ge both have odd parity.

IQP: Classical strategy [SB '08]

Quantum: $\Pr[X^\intercal \cdot s = 0] \approx 0.85$
Classical: $\Pr[Y^\intercal \cdot s = 0] = 0.75$

Consider choosing random $d, e \xleftarrow{\$} \{0,1\}^n$, and letting

$$y = \sum_{\substack{p \in \mathrm{rows}(P) \\ p \cdot d = p \cdot e = 1}} p$$

Then:

$$y \cdot s = (Gd) \cdot (Ge) \pmod 2$$

Fact: $(Gd) \cdot (Ge) = 1$ iff Gd, Ge both have odd parity.

Thus $y \cdot s = 0$ with probability 3/4!

IQP: Can we do better classically? [GDKM ’19 arXiv:1912.05547]

Key: Correlate samples to attack the key s

IQP: Can we do better classically? [GDKM ’19 arXiv:1912.05547]

Key: Correlate samples to attack the key $\boldsymbol{s}$

Consider choosing one random $\boldsymbol{d} \xleftarrow{\$} \{0,1\}^n$, held constant over many different $\boldsymbol{e}_i \xleftarrow{\$} \{0,1\}^n$

$$\boldsymbol{y}_i = \sum_{\substack{p \in \mathrm{rows}(P) \\ \boldsymbol{p}\cdot\boldsymbol{d} = \boldsymbol{p}\cdot\boldsymbol{e}_i = 1}} \boldsymbol{p}$$

$\boldsymbol{y}_i \cdot s = 1$ iff $G\boldsymbol{d}$, $G\boldsymbol{e}_i$ both have odd parity.

IQP: Can we do better classically? [GDKM '19 arXiv:1912.05547]

Key: Correlate samples to attack the key $\mathbf{s}$

Consider choosing one random $\mathbf{d} \xleftarrow{\$} \{0,1\}^n$, held constant over many different $\mathbf{e}_i \xleftarrow{\$} \{0,1\}^n$

$$\mathbf{y}_i = \sum_{\substack{p \in \mathrm{rows}(P) \\ \mathbf{p}\cdot\mathbf{d} = \mathbf{p}\cdot\mathbf{e}_i = 1}} \mathbf{p}$$

$\mathbf{y}_i \cdot s = 1$ iff $G\mathbf{d}$, $G\mathbf{e}_i$ both have odd parity.

$G\mathbf{d}$ has even parity $\Rightarrow$ *all* $\mathbf{y}_i \cdot s = 0$

IQP: Can we do better classically? [GDKM '19 arXiv:1912.05547]

Key: Correlate samples to attack the key $\mathbf{s}$

Consider choosing one random $\mathbf{d} \xleftarrow{\$} \{0,1\}^n$, held constant over many different $\mathbf{e}_i \xleftarrow{\$} \{0,1\}^n$

$$\mathbf{y}_i = \sum_{\substack{p \in \mathrm{rows}(P) \\ \mathbf{p}\cdot\mathbf{d} = \mathbf{p}\cdot\mathbf{e}_i = 1}} \mathbf{p}$$

$\mathbf{y}_i \cdot s = 1$ iff $G\mathbf{d}$, $G\mathbf{e}_i$ both have odd parity.

$G\mathbf{d}$ has even parity $\Rightarrow$ *all* $\mathbf{y}_i \cdot s = 0$

Let y_i form rows of a matrix M, such that $M\mathbf{s} = \mathbf{0}$

IQP: Can we do better classically? [GDKM '19 arXiv:1912.05547]

Key: Correlate samples to attack the key $\boldsymbol{s}$

Consider choosing one random $\boldsymbol{d} \xleftarrow{\$} \{0,1\}^n$, held constant over many different $\boldsymbol{e}_i \xleftarrow{\$} \{0,1\}^n$

$$\boldsymbol{y}_i = \sum_{\substack{p \in \mathrm{rows}(P) \\ \boldsymbol{p}\cdot\boldsymbol{d} = \boldsymbol{p}\cdot\boldsymbol{e}_i = 1}} \boldsymbol{p}$$

$\boldsymbol{y}_i \cdot s = 1$ iff $G\boldsymbol{d}$, $G\boldsymbol{e}_i$ both have odd parity.

$G\boldsymbol{d}$ has even parity $\Rightarrow$ *all* $\boldsymbol{y}_i \cdot s = 0$

Let y_i form rows of a matrix M, such that $M\boldsymbol{s} = \boldsymbol{0}$

Can solve for $\boldsymbol{s}$! … If M has high rank.

IQP: Can we do better classically? [GDKM '19 arXiv:1912.05547]

Key: Correlate samples to attack the key $\boldsymbol{s}$

Consider choosing one random $\boldsymbol{d} \xleftarrow{\$} \{0,1\}^n$, held constant over many different $\boldsymbol{e}_i \xleftarrow{\$} \{0,1\}^n$

$$\boldsymbol{y}_i = \sum_{\substack{p \in \mathrm{rows}(P) \\ \boldsymbol{p}\cdot\boldsymbol{d} = \boldsymbol{p}\cdot\boldsymbol{e}_i = 1}} \boldsymbol{p}$$

$\boldsymbol{y}_i \cdot s = 1$ iff $G\boldsymbol{d}$, $G\boldsymbol{e}_i$ both have odd parity.

$G\boldsymbol{d}$ has even parity $\Rightarrow$ *all* $\boldsymbol{y}_i \cdot s = 0$

Let y_i form rows of a matrix M, such that $M\boldsymbol{s} = \boldsymbol{0}$

Can solve for $\boldsymbol{s}$! ... If M has high rank. Empirically it does!

IQP: can it be fixed?

- Attack relies on properties of QR code

IQP: can it be fixed?

- Attack relies on properties of QR code
- Could pick a different G for which this attack would not succeed?

IQP: can it be fixed?

- Attack relies on properties of QR code
- Could pick a different G for which this attack would not succeed?
- Ultimately, would like to rely on standard cryptographic assumptions…

NISQ verifiable quantum advantage

Sampling problems

e.g. random circuits, Boson sampling, ...

- ✓ NISQ feasible
- ✗ Efficiently verifiable

Number theory problems

e.g. factoring, discrete logarithm, ...

- ✗ NISQ feasible
- ✓ Efficiently verifiable

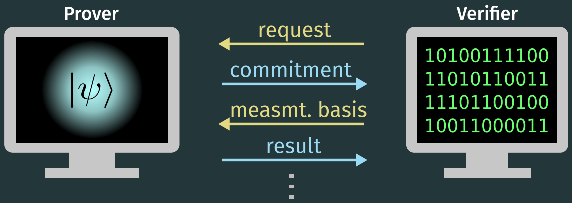
add structure

make less costly

???

- ✓ NISQ feasible
- ✓ Efficiently verifiable

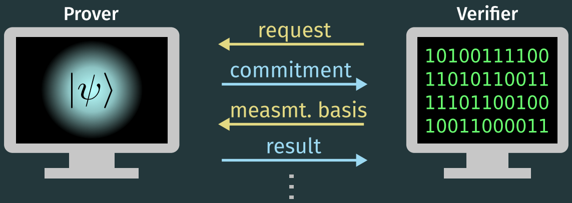
Interactive proofs of quantumness



Round 1: Prover commits to a specific quantum state

Round 2+: Verifier asks for measurement in specific basis

Interactive proofs of quantumness



Round 1: Prover commits to a specific quantum state

Round 2+: Verifier asks for measurement in specific basis

By randomizing choice of basis and repeating interaction, can ensure prover would respond correctly in *any* basis

Brakerski, Christiano, Mahadev, Vidick, Vazirani '18 (arXiv:1804.00640).

Can be extended to verify arbitrary quantum computations! (arXiv:1804.01082)

State commitment (round 1): trapdoor claw-free functions

How does the prover commit to a state?

Consider a **2-to-1** collision-resistant (claw-free) function f.

State commitment (round 1): trapdoor claw-free functions

How does the prover commit to a state?

Consider a **2-to-1** collision-resistant (claw-free) function f.

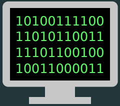
Evaluate f on uniform superposition $\sum_x \|x\rangle \|f(x)\rangle$	$\xleftarrow{\quad f \quad}$	Pick 2-to-1 function f
Measure 2^{nd} register as y	$\xrightarrow{\quad y \quad}$	Store y as commitment

Prover has committed to the state $(|x_0\rangle + |x_1\rangle)\,|y\rangle$

LWE protocol

Prover

Verifier

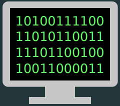


Pick trapdoor claw-free function f $\xleftarrow{f}$ Evaluate f on uniform superposition: $\sum_x |x\rangle |f(x)\rangle$

Measure 2^{nd} register as y $\xrightarrow{y}$ Compute x_0, x_1 from y using trapdoor

Brakerski, Christiano, Mahadev, Vidick, Vazirani '18 (arXiv:1804.00640)

LWE protocol

Prover		Verifier
		
Evaluate f on uniform superposition: $\sum_x \|x\rangle\,\|f(x)\rangle$	$\xleftarrow{\ f\ }$	Pick trapdoor claw-free function f
Measure 2nd register as y	$\xrightarrow{\ y\ }$	Compute x_0, x_1 from y using trapdoor
	$\xleftarrow{\text{basis}}$	Pick standard or Hadamard basis
Measure qubits of $\|x_0\rangle + \|x_1\rangle$ in given basis		
	$\xrightarrow{\text{result}}$	Validate result against x_0, x_1

Brakerski, Christiano, Mahadev, Vidick, Vazirani '18 (arXiv:1804.00640)

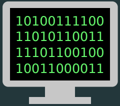
LWE protocol

Prover		Verifier
		
Evaluate f on uniform superposition: $\sum_x \lvert x\rangle \lvert f(x)\rangle$	$\xleftarrow{\quad f \quad}$	Pick trapdoor claw-free function f
Measure 2nd register as y	$\xrightarrow{\quad y \quad}$	Compute x_0, x_1 from y using trapdoor
	$\xleftarrow{\text{basis}}$	Pick standard or Hadamard basis
Measure qubits of $\lvert x_0\rangle + \lvert x_1\rangle$ in given basis		
	$\xrightarrow{\text{result}}$	Validate result against x_0, x_1

Subtlety: claw-free does *not* imply hardness of generating measurement outcomes!

Brakerski, Christiano, Mahadev, Vidick, Vazirani '18 (arXiv:1804.00640)

LWE protocol

Prover		Verifier
		
Evaluate f on uniform superposition: $\sum_x \|x\rangle \|f(x)\rangle$	$\xleftarrow{\quad f \quad}$	Pick trapdoor claw-free function f
Measure 2nd register as y	$\xrightarrow{\quad y \quad}$	Compute x_0, x_1 from y using trapdoor
	$\xleftarrow{\text{basis}}$	Pick standard or Hadamard basis
Measure qubits of $\|x_0\rangle + \|x_1\rangle$ in given basis		
	$\xrightarrow{\text{result}}$	Validate result against x_0, x_1

Subtlety: claw-free does *not* imply hardness of generating measurement outcomes!
Learning-with-Errors TCF has adaptive hardcore bit

Brakerski, Christiano, Mahadev, Vidick, Vazirani '18 (arXiv:1804.00640)

Trapdoor claw-free functions

TCF	Trapdoor	Claw-free	Adaptive hard-core bit
LWE [1]	✓	✓	✓
$x^2 \bmod N$ [3]	✓	✓	✗
Ring-LWE [2]	✓	✓	✗
Diffie-Hellman [3]	✓	✓	✗

[1] Brakerski, Christiano, Mahadev, Vazirani, Vidick '18 (arXiv:1804.00640)

[2] Brakerski, Koppula, Vazirani, Vidick '20 (arXiv:2005.04826)

[3] GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

Trapdoor claw-free functions

TCF	Trapdoor	Claw-free	Adaptive hard-core bit
LWE [1]	✓	✓	✓
$x^2 \bmod N$ [3]	✓	✓	✗
Ring-LWE [2]	✓	✓	✗
Diffie-Hellman [3]	✓	✓	✗

BKVV '20 [2]: Non-interactive protocol without adaptive hardcore bit, in random oracle model

[1] Brakerski, Christiano, Mahadev, Vazirani, Vidick '18 (arXiv:1804.00640)

[2] Brakerski, Koppula, Vazirani, Vidick '20 (arXiv:2005.04826)

[3] GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

Trapdoor claw-free functions

TCF	Trapdoor	Claw-free	Adaptive hard-core bit
LWE [1]	✓	✓	✓
$x^2 \bmod N$ [3]	✓	✓	✗
Ring-LWE [2]	✓	✓	✗
Diffie-Hellman [3]	✓	✓	✗

BKVV ’20 [2]: Non-interactive protocol without adaptive hardcore bit, in random oracle model

Can we remove AHCB in the standard model of cryptography?

[1] Brakerski, Christiano, Mahadev, Vazirani, Vidick ’18 (arXiv:1804.00640)

[2] Brakerski, Koppula, Vazirani, Vidick ’20 (arXiv:2005.04826)

[3] GDKM, Choi, Vazirani, Yao ’21 (arXiv:2104.00687)

LWE protocol

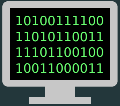
Prover		Verifier
		
Evaluate f on uniform superposition: $\sum_x \|x\rangle \|f(x)\rangle$	$\xleftarrow{\quad f \quad}$	Pick trapdoor claw-free function f
Measure 2nd register as y	$\xrightarrow{\quad y \quad}$	Compute x_0, x_1 from y using trapdoor
Measure qubits of $\|x_0\rangle + \|x_1\rangle$ in given basis	$\xleftarrow{\text{basis}}$	Pick standard or Hadamard basis
	$\xrightarrow{\text{result}}$	Validate result against x_0, x_1

Replace Hadamard basis measurement with “1-player CHSH”

Brakerski, Christiano, Mahadev, Vidick, Vazirani ’18 (arXiv:1804.00640)

Interactive measurement: computational Bell test

Replace Hadamard basis measurement with two-step process: "condense" x_0, x_1 into a single qubit, and then do a "Bell test."



$\vdots \qquad \vdots \qquad \vdots$

$|x_0\rangle\,|x_0 \cdot r\rangle + |x_1\rangle\,|x_1 \cdot r\rangle \quad \xleftarrow{\quad r \quad} \quad$ Pick random bitstring r

Measure all but ancilla in Hadamard basis $\quad \xrightarrow{\quad d \quad}$

GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

Interactive measurement: computational Bell test

Replace Hadamard basis measurement with two-step process: "condense" x_0, x_1 into a single qubit, and then do a "Bell test."

$\vdots \qquad \vdots \qquad \vdots$

$|x_0\rangle |x_0 \cdot r\rangle + |x_1\rangle |x_1 \cdot r\rangle \quad \xleftarrow{\quad r \quad} \quad$ Pick random bitstring r

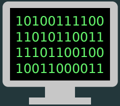
Measure all but ancilla in Hadamard basis $\quad \xrightarrow{\quad d \quad}$

Now single-qubit state: $|0\rangle$ or $|1\rangle$ if $x_0 \cdot r = x_1 \cdot r$, otherwise $|+\rangle$ or $|-\rangle$.

GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

Interactive measurement: computational Bell test

Replace Hadamard basis measurement with two-step process: "condense" x_0, x_1 into a single qubit, and then do a "Bell test."



$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

$|x_0\rangle\,|x_0 \cdot r\rangle + |x_1\rangle\,|x_1 \cdot r\rangle$ $\quad \xleftarrow{\quad r \quad}$ $\quad$ Pick random bitstring r

Measure all but ancilla in Hadamard basis $\quad \xrightarrow{\quad d \quad}$

Now single-qubit state: $|0\rangle$ or $|1\rangle$ if $x_0 \cdot r = x_1 \cdot r$, otherwise $|+\rangle$ or $|-\rangle$.
Polarization hidden via:

Cryptographic secret (here) $\Leftrightarrow$ Non-communication (Bell test)

GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

Interactive measurement: computational Bell test

Replace Hadamard basis measurement with two-step process: "condense" x_0, x_1 into a single qubit, and then do a "Bell test."

$\vdots$ $\qquad$ $\vdots$ $\qquad$ $\vdots$

$\lvert x_0\rangle \lvert x_0 \cdot r\rangle + \lvert x_1\rangle \lvert x_1 \cdot r\rangle$	$\xleftarrow{\quad r \quad}$	Pick random bitstring r
Measure all but ancilla in Hadamard basis	$\xrightarrow{\quad d \quad}$	
Measure qubit in basis	$\xleftarrow{\text{basis}}$	Pick $(Z + X)$ or $(Z - X)$ basis
	$\xrightarrow{\text{result}}$	Validate against r, x_0, x_1, d

GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

Computational Bell test: classical bound

Run protocol many times, collect statistics.

p_s: Success rate for standard basis measurement.

p_{CHSH}: Success rate when performing CHSH-type measurement.

Computational Bell test: classical bound

Run protocol many times, collect statistics.

p_s: Success rate for standard basis measurement.

p_{CHSH}: Success rate when performing CHSH-type measurement.

Under assumption of claw-free function:

Classical bound: $p_s + 4p_{\text{CHSH}} - 4 < \text{negl}(n)$

Computational Bell test: classical bound

Run protocol many times, collect statistics.

p_s: Success rate for standard basis measurement.

p_{CHSH}: Success rate when performing CHSH-type measurement.

Under assumption of claw-free function:

Classical bound: $p_s + 4p_{\text{CHSH}} - 4 < \text{negl}(n)$

Ideal quantum: $p_s = 1, p_{\text{CHSH}} = \cos^2(\pi/8)$

Computational Bell test: classical bound

Run protocol many times, collect statistics.

p_s: Success rate for standard basis measurement.

p_{CHSH}: Success rate when performing CHSH-type measurement.

Under assumption of claw-free function:

Classical bound: $p_s + 4p_{\text{CHSH}} - 4 < \text{negl}(n)$

Ideal quantum: $p_s = 1, p_{\text{CHSH}} = \cos^2(\pi/8)$

$p_s + 4p_{\text{CHSH}} - 4 = \sqrt{2} - 1 \approx \mathbf{0.414}$

Computational Bell test: classical bound

Run protocol many times, collect statistics.

p_s: Success rate for standard basis measurement.

p_{CHSH}: Success rate when performing CHSH-type measurement.

Under assumption of claw-free function:

Classical bound: $p_s + 4p_{\text{CHSH}} - 4 < \text{negl}(n)$

Ideal quantum: $p_s = 1, p_{\text{CHSH}} = \cos^2(\pi/8)$

$p_s + 4p_{\text{CHSH}} - 4 = \sqrt{2} - 1 \approx \mathbf{0.414}$

Note: Let $p_s = 1$. Then for p_{CHSH}:

Classical bound 75%, ideal quantum $\sim$ 85%. Same as regular CHSH!

GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

Challenges for implementation

- Partial measurement

Challenges for implementation

- Partial measurement
    - Required for multi-round classical interaction

Challenges for implementation

- Partial measurement
    - Required for multi-round classical interaction
- Fidelity requirement

Challenges for implementation

- Partial measurement
    - Required for multi-round classical interaction
- Fidelity requirement
    - High fidelity needed to pass classical bound

Challenges for implementation

- Partial measurement
  - Required for multi-round classical interaction
- Fidelity requirement
  - High fidelity needed to pass classical bound
- Circuit sizes

Challenges for implementation

- Partial measurement
    - Required for multi-round classical interaction
- Fidelity requirement
    - High fidelity needed to pass classical bound
- Circuit sizes
    - Need to implement public-key crypto. on a superposition

Partial measurements in the lab

TIQI

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Prof. Christopher Monroe

Dr. Daiwei Zhu

Dr. Crystal Noel

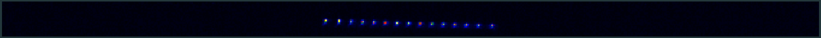
and others!

Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:

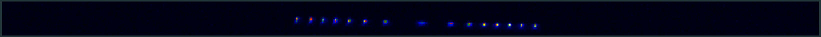


Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:



Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:

Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:

Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:

Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:

Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:

Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

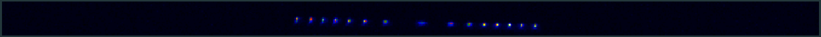
Partial measurement:

Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:

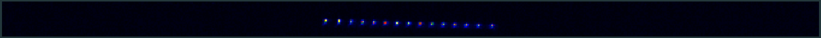


Partial measurements in the lab

Trapped Ion Quantum Information lab at U. Maryland

Working on demonstration of protocols in trapped ions!

Partial measurement:



Technique: postselection

How to deal with high fidelity requirement? Need $\sim 83\%$ fidelity in general to pass.

Technique: postselection

How to deal with high fidelity requirement? Need $\sim 83\%$ fidelity in general to pass.

Can show: a prover holding $(|x_0\rangle + |x_1\rangle)|y\rangle$ with ϵ phase coherence passes!

Technique: postselection

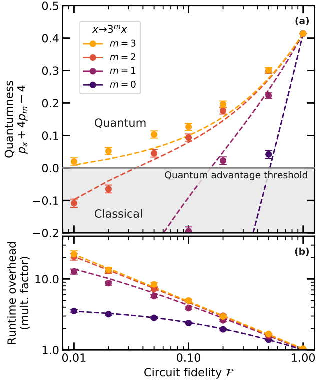
How to deal with high fidelity requirement? Need $\sim 83\%$ fidelity in general to pass.

Can show: a prover holding $(|x_0\rangle + |x_1\rangle)\,|y\rangle$ with ϵ phase coherence passes!

When we generate $\sum_x |x\rangle\,|f(x)\rangle$, **add redundancy to $f(x)$, for bit flip error detection!**

Technique: postselection

How to deal with high fidelity requirement? Need $\sim 83\%$ fidelity in general to pass.



Numerical results for $x^2 \bmod N$ with $\log N = 512$ bits.

Here: make transformation $x^2 \bmod N \Rightarrow (kx)^2 \bmod k^2 N$

Improving circuit sizes

Most demanding step in all these protocols: evaluating TCF

$$\mathcal{U}_f \left|x\right\rangle \left|0^{\otimes n}\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$$

Improving circuit sizes

Most demanding step in all these protocols: evaluating TCF

$$\mathcal{U}_f \left|x\right\rangle \left|0^{\otimes n}\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$$

Getting rid of adaptive hardcore bit helps!

$x^2 \mod N$ and **Ring-LWE** have classical circuits as fast as $\mathcal{O}(n \log n)$...

Improving circuit sizes

Most demanding step in all these protocols: evaluating TCF

$$\mathcal{U}_f |x\rangle |0^{\otimes n}\rangle = |x\rangle |f(x)\rangle$$

Getting rid of adaptive hardcore bit helps!

$x^2 \mod N$ and **Ring-LWE** have classical circuits as fast as $\mathcal{O}(n \log n)$...

but they are recursive and hard to make reversible.

Improving circuit sizes

Most demanding step in all these protocols: evaluating TCF

$$\mathcal{U}_f \left|x\right\rangle \left|0^{\otimes n}\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$$

Getting rid of adaptive hardcore bit helps!

$x^2 \mod N$ and Ring-LWE have classical circuits as fast as $\mathcal{O}(n \log n)$...

but they are recursive and hard to make reversible.

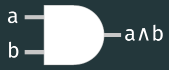
Protocol allows us to make circuits irreversible!

Technique: taking out the garbage

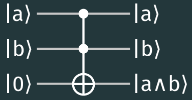
Goal: $\mathcal{U}_f \left|x\right\rangle \left|0^{\otimes n}\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$

When converting classical circuits to quantum:

Garbage bits: extra entangled outputs due to unitarity



Classical AND



Quantum AND (Toffoli)

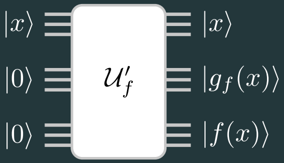
Technique: taking out the garbage

Goal: $\mathcal{U}_f |x\rangle |0^{\otimes n}\rangle = |x\rangle |f(x)\rangle$

When converting classical circuits to quantum:

Garbage bits: extra entangled outputs due to unitarity

Let $\mathcal{U}_f'$ be a unitary generating garbage bits $g_f(x)$:



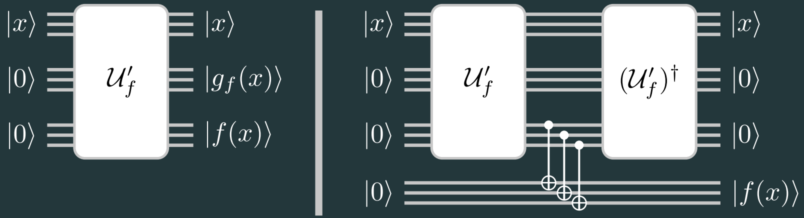
Technique: taking out the garbage

Goal: $\mathcal{U}_f \left|x\right\rangle \left|0^{\otimes n}\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$

When converting classical circuits to quantum:

Garbage bits: extra entangled outputs due to unitarity

Let $\mathcal{U}_f'$ be a unitary generating garbage bits $g_f(x)$:



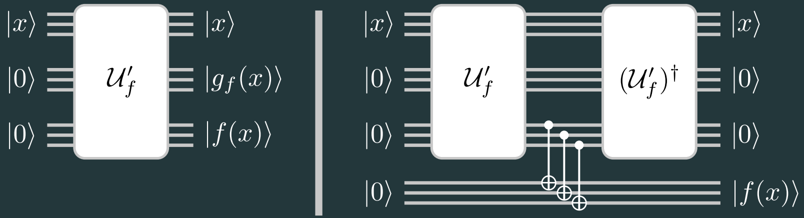
Technique: taking out the garbage

Goal: $\mathcal{U}_f \left|x\right\rangle \left|0^{\otimes n}\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$

When converting classical circuits to quantum:

Garbage bits: extra entangled outputs due to unitarity

Let $\mathcal{U}_f'$ be a unitary generating garbage bits $g_f(x)$:



Lots of time and space overhead!

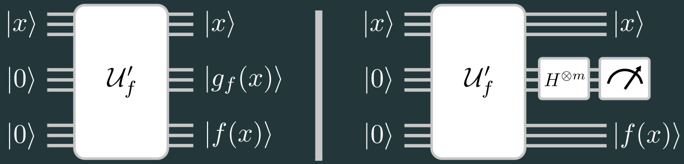
Technique: taking out the garbage

Goal: $\mathcal{U}_f \left|x\right\rangle \left|0^{\otimes n}\right\rangle = \left|x\right\rangle \left|f(x)\right\rangle$

When converting classical circuits to quantum:

Garbage bits: extra entangled outputs due to unitarity

Let $\mathcal{U}_f'$ be a unitary generating garbage bits $g_f(x)$:



Can we “measure them away” instead?

Technique: taking out the garbage

Measure garbage bits $g_f(x)$ in Hadamard basis, get some string h.
End up with state:

$$\sum_x (-1)^{h \cdot g_f(x)} |x\rangle |f(x)\rangle$$

Technique: taking out the garbage

Measure garbage bits $g_f(x)$ in Hadamard basis, get some string h.
End up with state:

$$\sum_x (-1)^{h \cdot g_f(x)} |x\rangle |f(x)\rangle$$

In general useless: unique phase $(-1)^{h \cdot g_f(x)}$ on every term.

Technique: taking out the garbage

Measure garbage bits $g_f(x)$ in Hadamard basis, get some string h.
End up with state:

$$\sum_x (-1)^{h \cdot g_f(x)} |x\rangle |f(x)\rangle$$

In general useless: unique phase $(-1)^{h \cdot g_f(x)}$ on every term.

But after collapsing onto a single output:

$$\left[(-1)^{h \cdot g_f(x_0)} |x_0\rangle + (-1)^{h \cdot g_f(x_1)} |x_1\rangle\right] |y\rangle$$

Technique: taking out the garbage

Measure garbage bits $g_f(x)$ in Hadamard basis, get some string h.
End up with state:

$$\sum_x (-1)^{h \cdot g_f(x)} |x\rangle |f(x)\rangle$$

In general useless: unique phase $(-1)^{h \cdot g_f(x)}$ on every term.

But after collapsing onto a single output:

$$\left[(-1)^{h \cdot g_f(x_0)} |x_0\rangle + (-1)^{h \cdot g_f(x_1)} |x_1\rangle\right] |y\rangle$$

Verifier can efficiently compute $g_f(\cdot)$ for these two terms!

Technique: taking out the garbage

Measure garbage bits $g_f(x)$ in Hadamard basis, get some string h.
End up with state:

$$\sum_x (-1)^{h \cdot g_f(x)} |x\rangle |f(x)\rangle$$

In general useless: unique phase $(-1)^{h \cdot g_f(x)}$ on every term.

But after collapsing onto a single output:

$$\left[(-1)^{h \cdot g_f(x_0)} |x_0\rangle + (-1)^{h \cdot g_f(x_1)} |x_1\rangle\right] |y\rangle$$

Verifier can efficiently compute $g_f(\cdot)$ for these two terms!

Can directly convert classical circuits to quantum!

Technique: taking out the garbage

Measure garbage bits $g_f(x)$ in Hadamard basis, get some string h.
End up with state:

$$\sum_x (-1)^{h \cdot g_f(x)} |x\rangle |f(x)\rangle$$

In general useless: unique phase $(-1)^{h \cdot g_f(x)}$ on every term.

But after collapsing onto a single output:

$$\left[(-1)^{h \cdot g_f(x_0)} |x_0\rangle + (-1)^{h \cdot g_f(x_1)} |x_1\rangle\right] |y\rangle$$

Verifier can efficiently compute $g_f(\cdot)$ for these two terms!

Can directly convert classical circuits to quantum!
1024-bit $x^2 \bmod N$ costs only 10^6 Toffoli gates.

Paths forward

Bottleneck: Evaluating TCF on quantum superposition

Paths forward

Bottleneck: Evaluating TCF on quantum superposition

"In the box" ideas (not necessarily bad):

- Find more efficient TCFs

Paths forward

Bottleneck: Evaluating TCF on quantum superposition

"In the box" ideas (not necessarily bad):

- Find more efficient TCFs
- Better quantum circuits for TCFs

Paths forward

Bottleneck: Evaluating TCF on quantum superposition

"In the box" ideas (not necessarily bad):

- Find more efficient TCFs
- Better quantum circuits for TCFs
- … public-key cryptography is just slow

Paths forward

Bottleneck: Evaluating TCF on quantum superposition

“In the box” ideas (not necessarily bad):

- Find more efficient TCFs
- Better quantum circuits for TCFs
- … public-key cryptography is just slow

Paths forward

Bottleneck: Evaluating TCF on quantum superposition

"In the box" ideas (not necessarily bad):

- Find more efficient TCFs
- Better quantum circuits for TCFs
- … public-key cryptography is just slow

"Box-adjacent" ideas:

- Explore other protocols (fix IQP and make it fast?)

Paths forward

Bottleneck: Evaluating TCF on quantum superposition

"In the box" ideas (not necessarily bad):

- Find more efficient TCFs
- Better quantum circuits for TCFs
- … public-key cryptography is just slow

"Box-adjacent" ideas:

- Explore other protocols (fix IQP and make it fast?)
- Remove trapdoor—hash-based cryptography?

Paths forward

Bottleneck: Evaluating TCF on quantum superposition

"In the box" ideas (not necessarily bad):

- Find more efficient TCFs
- Better quantum circuits for TCFs
- … public-key cryptography is just slow

"Box-adjacent" ideas:

- Explore other protocols (fix IQP and make it fast?)
- Remove trapdoor—hash-based cryptography?

Paths forward

Bottleneck: Evaluating TCF on quantum superposition

"In the box" ideas (not necessarily bad):

- Find more efficient TCFs
- Better quantum circuits for TCFs
- … public-key cryptography is just slow

"Box-adjacent" ideas:

- Explore other protocols (fix IQP and make it fast?)
- Remove trapdoor—hash-based cryptography?

Way outside the box?

Backup!

TCF constructions

TCF	A.H.C.B.	Gate count	n for hardness
LWE [1]	✓	$\mathcal{O}(n^2 \log^2 n)$	10^4
Ring-LWE [2]	✗	$\mathcal{O}(n \log^2 n)$	10^3
$x^2 \bmod N$ [3]	✗	$\mathcal{O}(n \log n)$	10^3
DDH [3]	✗	$\mathcal{O}(n^3 \log^2 n)$	10^2

A.H.C.B. = "adaptive hard core bit"

[1] Brakerski, Christiano, Mahadev, Vidick, Vazirani '18 (arXiv:1804.00640)

[2] Brakerski, Koppula, Vazirani, Vidick '20 (arXiv:2005.04826)

[3] GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

TCF constructions

TCF	A.H.C.B.	Gate count	n for hardness
LWE [1]	✓	$\mathcal{O}(n^2 \log^2 n)$	10^4
Ring-LWE [2]	✗	$\mathcal{O}(n \log^2 n)$	10^3
$x^2 \bmod N$ [3]	✗	$\mathcal{O}(n \log n)$	10^3
DDH [3]	✗	$\mathcal{O}(n^3 \log^2 n)$	10^2

A.H.C.B. = ”adaptive hard core bit”

Remarks:

- Removing adaptive hardcore bit requirement helps!

[1] Brakerski, Christiano, Mahadev, Vidick, Vazirani ’18 (arXiv:1804.00640)

[2] Brakerski, Koppula, Vazirani, Vidick ’20 (arXiv:2005.04826)

[3] GDKM, Choi, Vazirani, Yao ’21 (arXiv:2104.00687)

TCF constructions

TCF	A.H.C.B.	Gate count	n for hardness
LWE [1]	✓	$\mathcal{O}(n^2 \log^2 n)$	10^4
Ring-LWE [2]	✗	$\mathcal{O}(n \log^2 n)$	10^3
$x^2 \bmod N$ [3]	✗	$\mathcal{O}(n \log n)$	10^3
DDH [3]	✗	$\mathcal{O}(n^3 \log^2 n)$	10^2

A.H.C.B. = "adaptive hard core bit"

Remarks:

- Removing adaptive hardcore bit requirement helps!
- Can't just plug in n—constant factors

[1] Brakerski, Christiano, Mahadev, Vidick, Vazirani '18 (arXiv:1804.00640)

[2] Brakerski, Koppula, Vazirani, Vidick '20 (arXiv:2005.04826)

[3] GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

$x^2 \bmod N$

$$y = x^2 \bmod N \text{ with } N = pq$$

Each y has 4 roots $(x_0, x_1, -x_0, -x_1)$.

$x^2 \bmod N$

$$y = x^2 \bmod N \text{ with } N = pq$$

Each y has 4 roots $(x_0, x_1, -x_0, -x_1)$. Set domain to $[0, N/2]$ to make it 2-to-1

$x^2 \bmod N$

$$y = x^2 \bmod N \text{ with } N = pq$$

Each y has 4 roots $(x_0, x_1, -x_0, -x_1)$. Set domain to $[0, N/2]$ to make it 2-to-1

- Finding a claw as hard as factoring N

$x^2 \bmod N$

$$y = x^2 \bmod N \text{ with } N = pq$$

Each y has 4 roots $(x_0, x_1, -x_0, -x_1)$. Set domain to $[0, N/2]$ to make it 2-to-1

- Finding a claw as hard as factoring N
- Features:
  - Simple to implement, asymptotically fast algorithms
  - Classical hardness in practice extremely well studied

$x^2 \bmod N$

$$y = x^2 \bmod N \text{ with } N = pq$$

Each y has 4 roots $(x_0, x_1, -x_0, -x_1)$. Set domain to $[0, N/2]$ to make it 2-to-1

- Finding a claw as hard as factoring N
- Features:
  - Simple to implement, asymptotically fast algorithms
  - Classical hardness in practice extremely well studied
- $\mathcal{O}(n \log n \log \log n)$ Schonhage-Strassen multiplication seems out of reach, but

$x^2 \bmod N$

$$y = x^2 \bmod N \text{ with } N = pq$$

Each y has 4 roots $(x_0, x_1, -x_0, -x_1)$. Set domain to $[0, N/2]$ to make it 2-to-1

- Finding a claw as hard as factoring N
- Features:
  - Simple to implement, asymptotically fast algorithms
  - Classical hardness in practice extremely well studied
- $\mathcal{O}(n \log n \log \log n)$ Schonhage-Strassen multiplication seems out of reach, but
- $\mathcal{O}(n^{1.58})$ Karatsuba mult. beats naive $\mathcal{O}(n^2)$ alg. at $n \sim 100$ (much earlier than in the classical case!)

$x^2 \bmod N$

$$y = x^2 \bmod N \text{ with } N = pq$$

Each y has 4 roots $(x_0, x_1, -x_0, -x_1)$. Set domain to $[0, N/2]$ to make it 2-to-1

- Finding a claw as hard as factoring N
- Features:
  - Simple to implement, asymptotically fast algorithms
  - Classical hardness in practice extremely well studied
- $\mathcal{O}(n \log n \log \log n)$ Schonhage-Strassen multiplication seems out of reach, but
- $\mathcal{O}(n^{1.58})$ Karatsuba mult. beats naive $\mathcal{O}(n^2)$ alg. at $n \sim 100$ (much earlier than in the classical case!)

Q. advantage in 10^6 Toffoli gates

Trapdoor from Decisional Diffie-Hellman (DDH)

Trapdoor functions from DDH [1, 2]: **linear algebra in the exponent**

Trapdoor from Decisional Diffie-Hellman (DDH)

Trapdoor functions from DDH [1, 2]: **linear algebra in the exponent**

$\mathsf{Gen}(1^\lambda)$

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g

[1] Peikert, Waters. "Lossy trapdoor functions and their applications" (2008)

[2] Freeman, Goldreich, Klitz, Rosen, Segev. "More constructions of lossy and correlation-secure trapdoor functions" (2010)

Trapdoor from Decisional Diffie-Hellman (DDH)

Trapdoor functions from DDH [1, 2]: **linear algebra in the exponent**

$\text{Gen}(1^\lambda)$

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $M \in \mathbb{Z}_q^{k \times k}$ for $k > \log q$

[1] Peikert, Waters. "Lossy trapdoor functions and their applications" (2008)

[2] Freeman, Goldreich, Klitz, Rosen, Segev. "More constructions of lossy and correlation-secure trapdoor functions" (2010)

Trapdoor from Decisional Diffie-Hellman (DDH)

Trapdoor functions from DDH [1, 2]: **linear algebra in the exponent**

Gen(1^λ)

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $M \in \mathbb{Z}_q^{k \times k}$ for $k > \log q$
3. Compute $g^M = (g^{M_{ij}}) \in \mathbb{G}^{k \times k}$

[1] Peikert, Waters. "Lossy trapdoor functions and their applications" (2008)

[2] Freeman, Goldreich, Klitz, Rosen, Segev. "More constructions of lossy and correlation-secure trapdoor functions" (2010)

Trapdoor from Decisional Diffie-Hellman (DDH)

Trapdoor functions from DDH [1, 2]: **linear algebra in the exponent**

$\mathrm{Gen}(1^\lambda)$

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $\boldsymbol{M} \in \mathbb{Z}_q^{k \times k}$ for $k > \log q$
3. Compute $g^{\boldsymbol{M}} = (g^{\boldsymbol{M}_{ij}}) \in \mathbb{G}^{k \times k}$
4. Return $pk = (g^M)$, $sk = (g, M)$

[1] Peikert, Waters. "Lossy trapdoor functions and their applications" (2008)

[2] Freeman, Goldreich, Klitz, Rosen, Segev. "More constructions of lossy and correlation-secure trapdoor functions" (2010)

Trapdoor from Decisional Diffie-Hellman (DDH)

Trapdoor functions from DDH [1, 2]: **linear algebra in the exponent**

$pk = (g^M)$, $sk = (g, M)$. On input $\boldsymbol{x} \in \{0, 1\}^k$:

Trapdoor from Decisional Diffie-Hellman (DDH)

Trapdoor functions from DDH [1, 2]: **linear algebra in the exponent**

$pk = (g^M)$, $sk = (g, M)$. On input $\mathbf{x} \in \{0,1\}^k$:

Evaluation: $f(x) = g^{\mathbf{Mx}}$

Trapdoor from Decisional Diffie-Hellman (DDH)

Trapdoor functions from DDH [1, 2]: **linear algebra in the exponent**

$pk = (g^{M})$, $sk = (g, M)$. On input $\mathbf{x} \in \{0,1\}^k$:

Evaluation: $f(x) = g^{\mathbf{Mx}}$

Inversion: $f^{-1}(f(x), M) = g^{\mathbf{M}^{-1}\mathbf{Mx}} = g^{\mathbf{x}}$

Easy to find $\mathbf{x}$ from $g^{\mathbf{x}}$ by brute force

---

Trapdoor from Decisional Diffie-Hellman (DDH)

Trapdoor functions from DDH [1, 2]: **linear algebra in the exponent**

$pk = (g^M)$, $sk = (g, M)$. On input $\mathbf{x} \in \{0,1\}^k$:

Evaluation: $f(x) = g^{\mathbf{Mx}}$

Inversion: $f^{-1}(f(x), M) = g^{\mathbf{M}^{-1}\mathbf{Mx}} = g^{\mathbf{x}}$

Easy to find $\mathbf{x}$ from $g^{\mathbf{x}}$ by brute force

---

Security proof: Given $g^{\mathbf{M}}$, DDH hides rank of $\mathbf{M}$. Inversion would imply algorithm to determine if $\mathbf{M}$ is full rank.

[1] Peikert, Waters. "Lossy trapdoor functions and their applications" (2008)

[2] Freeman, Goldreich, Klitz, Rosen, Segev. "More constructions of lossy and correlation-secure trapdoor functions" (2010)

TCF from DDH

$\mathsf{Gen}(1^\lambda)$

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g

TCF from DDH

Gen(1^λ)

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $M \in \mathbb{Z}_q^{k \times k}$ for $k > \log q$

TCF from DDH

$\mathsf{Gen}(1^\lambda)$

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $M \in \mathbb{Z}_q^{k \times k}$ for $k > \log q$
3. Compute $g^M = (g^{M_{ij}}) \in \mathbb{G}^{k \times k}$

TCF from DDH

$\mathsf{Gen}(1^\lambda)$

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $\boldsymbol{M} \in \mathbb{Z}_q^{k\times k}$ for $k > \log q$
3. Compute $g^{\boldsymbol{M}} = (g^{\boldsymbol{M}_{ij}}) \in \mathbb{G}^{k\times k}$
4. Choose $\boldsymbol{s} \in \{0,1\}^k$

TCF from DDH

Gen(1^λ)

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $\boldsymbol{M} \in \mathbb{Z}_q^{k\times k}$ for $k > \log q$
3. Compute $g^{\boldsymbol{M}} = (g^{\boldsymbol{M}_{ij}}) \in \mathbb{G}^{k\times k}$
4. Choose $\boldsymbol{s} \in \{0,1\}^k$
5. Return $pk = (g^{M}, g^{Ms})$, $sk = (g, M, s)$

TCF from DDH

Gen(1^λ)

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $\boldsymbol{M} \in \mathbb{Z}_q^{k \times k}$ for $k > \log q$
3. Compute $g^{\boldsymbol{M}} = (g^{\boldsymbol{M}_{ij}}) \in \mathbb{G}^{k \times k}$
4. Choose $\boldsymbol{s} \in \{0, 1\}^k$
5. Return $pk = (g^{\mathrm{M}}, g^{\mathrm{Ms}})$, $sk = (g, M, s)$

Gen(1^λ)

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $\boldsymbol{M} \in \mathbb{Z}_q^{k \times k}$ for $k > \log q$
3. Compute $g^{\boldsymbol{M}} = (g^{\boldsymbol{M}_{ij}}) \in \mathbb{G}^{k \times k}$
4. Choose $\boldsymbol{s} \in \{0,1\}^k$
5. Return $pk = (g^M, g^{Ms})$, $sk = (g, M, s)$

Evaluation:

Let $d \sim \mathcal{O}(k^2)$. Define two functions $f_b : \mathbb{Z}_d^k \to \mathbb{G}^k$:

$$f_0(x) = g^{Mx} \qquad\qquad f_1(x) = g^{Mx} g^{Ms} = g^{M(x+s)}$$

TCF from DDH

$\mathsf{Gen}(1^\lambda)$

1. Choose group $\mathbb{G}$ of order $q \sim \mathcal{O}(2^\lambda)$, and generator g
2. Choose random invertible $\boldsymbol{M} \in \mathbb{Z}_q^{k \times k}$ for $k > \log q$
3. Compute $g^{\boldsymbol{M}} = (g^{\boldsymbol{M}_{ij}}) \in \mathbb{G}^{k \times k}$
4. Choose $\boldsymbol{s} \in \{0,1\}^k$
5. Return $pk = (g^M, g^{Ms})$, $sk = (g, M, s)$

Evaluation:

Let $d \sim \mathcal{O}(k^2)$. Define two functions $f_b : \mathbb{Z}_d^k \to \mathbb{G}^k$:

$$f_0(x) = g^{Mx} \qquad\qquad f_1(x) = g^{Mx} g^{Ms} = g^{M(x+s)}$$

Inversion: $f^{-1}(f_0(x), M) = g^{\boldsymbol{M}^{-1}\boldsymbol{M}x} = g^{x}$ (poly-time brute force)

GDKM, Choi, Vazirani, Yao '21 (arXiv:2104.00687)

TCF from DDH: does it help?

- Via elliptic curves, can significantly reduce space requirement

TCF from DDH: does it help?

- Via elliptic curves, can significantly reduce space requirement
- But quantum circuit for group operation is **complicated**

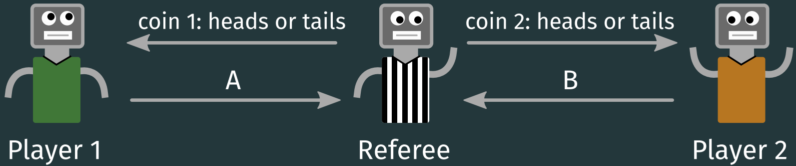
TCF from DDH: does it help?

- Via elliptic curves, can significantly reduce space requirement
- But quantum circuit for group operation is **complicated**
- Need to perform as many group operations as Shor's algorithm!

TCF from DDH: does it help?

- Via elliptic curves, can significantly reduce space requirement
- But quantum circuit for group operation is **complicated**
- Need to perform as many group operations as Shor's algorithm!
- Reversible Euclidean algorithm is hard, maybe irreversible optimization can help?

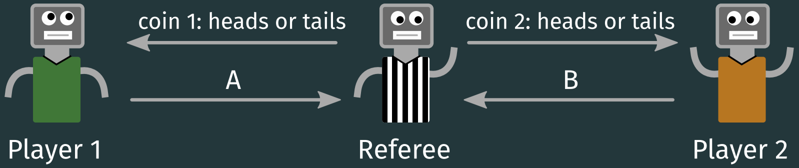
Two-player cooperative game.



If anyone receives tails, want $A = B$. If both get heads, want $A \neq B$.

The CHSH game (Bell test)

Two-player cooperative game.

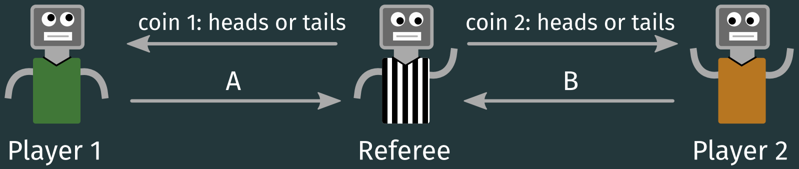


If anyone receives tails, want $A = B$. If both get heads, want $A \neq B$.

Two players sharing a Bell pair:

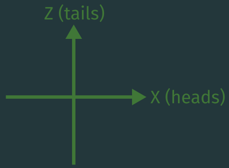
The CHSH game (Bell test)

Two-player cooperative game.



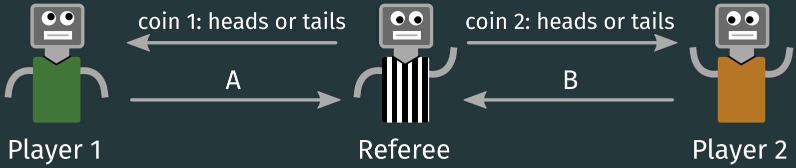
If anyone receives tails, want $A = B$. If both get heads, want $A \neq B$.

Two players sharing a Bell pair:



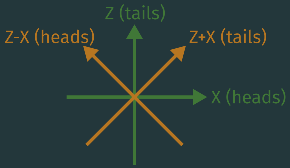
The CHSH game (Bell test)

Two-player cooperative game.



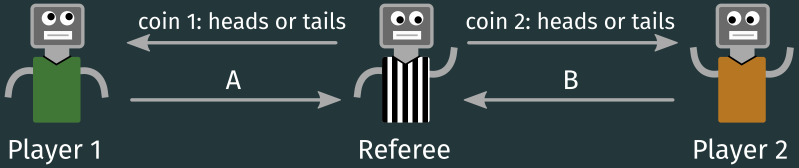
If anyone receives tails, want $A = B$. If both get heads, want $A \neq B$.

Two players sharing a Bell pair:



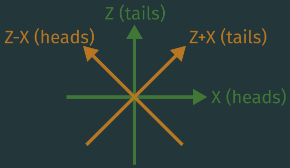
The CHSH game (Bell test)

Two-player cooperative game.



If anyone receives tails, want $A = B$. If both get heads, want $A \neq B$.

Two players sharing a Bell pair:



Quantum: $\cos^2(\pi/8) \approx 85\%$

Classical: 75%

Full protocol

Prover (quantum)

Verifier (classical)

Round 1

Verifier: 1. Sample $(f_i, t) \leftarrow \mathsf{Gen}(1^n)$

$\xleftarrow{f_i}$

Prover:

2. Generate state $\sum_{x=0}^{N/2} |x\rangle_{\mathsf{x}} |f_i(x)\rangle_{\mathsf{y}}$
3. Measure y register, yielding bitstring y
   State is now $(|x_0\rangle + |x_1\rangle)_{\mathsf{x}} |y\rangle_{\mathsf{y}}$;
   y register can be discarded

$\xrightarrow{y}$

Verifier: 4. Using trapdoor t compute x_0 and x_1

Verifier: 5. Randomly choose to request a preimage or continue

$\xleftarrow{\text{choice}}$

If preimage requested:

Prover: 6a. Projectively measure x register, yielding x

$\xrightarrow{x}$

Verifier: 7a. If $x \in \{x_0, x_1\}$ return Accept

Otherwise, continue:

Round 2

Verifier: 6b. Choose random bitstring r

$\xleftarrow{r}$

Prover:

7b. Add one ancilla b; use CNOTs to compute
$|r \cdot x_0\rangle_{\mathsf{b}} |x_0\rangle_{\mathsf{x}} + |r \cdot x_1\rangle_{\mathsf{b}} |x_1\rangle_{\mathsf{x}}$ where
$r \cdot x$ is bitwise inner product

8b. Measure x register in Hadamard basis,
yielding a string d. Discard x, state is now
$|\psi\rangle_{\mathsf{b}} \in \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$

$\xrightarrow{d}$

Verifier: 9b. Using r, x_0, x_1, d, determine $|\psi\rangle_{\mathsf{b}}$

Round 3

Verifier: 10b. Choose random $m \in \{\frac{\pi}{4}, -\frac{\pi}{4}\}$

$\xleftarrow{m}$

Prover: 11b. Measure ancilla b in the rotated basis

$$\left\{ \begin{matrix} \cos\left(\frac{m}{2}\right)|0\rangle + \sin\left(\frac{m}{2}\right)|1\rangle \\ \cos\left(\frac{m}{2}\right)|1\rangle - \sin\left(\frac{m}{2}\right)|0\rangle \end{matrix} \right\}, \text{ yielding a bit } b$$

$\xrightarrow{b}$

Verifier: 11b. If b was likely given $|\psi\rangle_{\mathsf{b}}$ return Accept