Fast quantum integer multiplication with almost no ancillas

Gregory D. Kahanamoku-Meyer

July 14, 2023

Motivation

Arithmetic on quantum computers: why do we care?

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OPEN

Classically verifiable quantum advantage from a computational Bell test

Gregory D. Kahanamoku-Meyer[1], Soonwon Choi[1], Umesh V. Vazirani[2] and Norman Y. Yao[1]

Existing experimental demonstrations of quantum computational advantage have had the limitation that verifying the correctness of the quantum device requires exponentially costly classical computations. Here we propose and analyse an interactive protocol for demonstrating quantum computational advantage, which is efficiently classically verifiable. Our protocol relies on a class of cryptographic tools called trapdoor claw-free functions. Although this type of function has been applied to quantum advantage protocols before, our protocol employs a surprising connection to Bell's inequality to avoid the need for a demanding cryptographic property called the adaptive hardcore bit, while maintaining essentially no increase in the quantum circuit complexity and no extra assumptions. Leveraging the relaxed cryptographic requirements of the protocol, we present two trapdoor claw-free function constructions, based on Rabin's function and the Diffie–Hellman problem, which have not been used in this context before. We also

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A Cryptographic Test of Quantumness and Certifiable Randomness from a Single Quantum Device

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PAUL CHRISTIANO, OpenAI, USA
URMILA MAHADEV, California Institute of Technology, USA
UMESH VAZIRANI, UC Berkeley, USA
THOMAS VIDICK, California Institute of Technology, USA

We consider a new model for the testing of untrusted quantum devices, consisting of a single polynomial time bounded quantum device interacting with a classical polynomial time verifier. In this model, we propose solutions to two tasks—a protocol for efficient classical verification that the untrusted device is "truly quantum" and a protocol for producing certifiable randomness from a single untrusted quantum device. Our solution relies on the existence of a new cryptographic primitive for constraining the power

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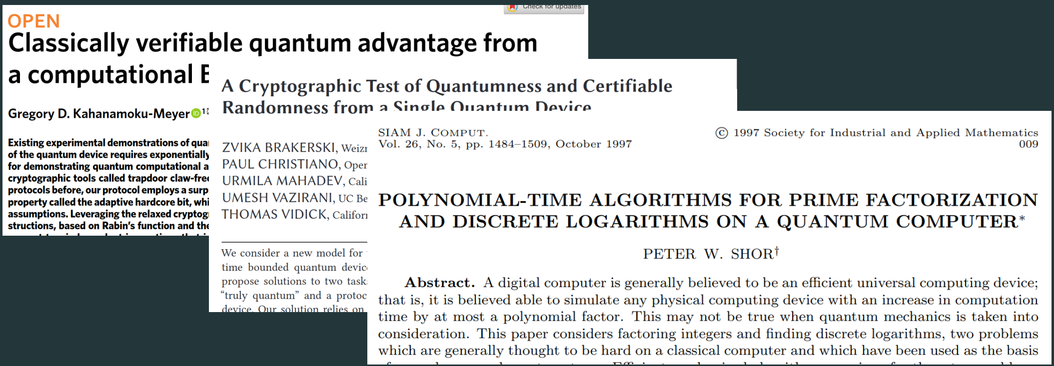
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POLYNOMIAL-TIME ALGORITHMS FOR PRIME FACTORIZATION AND DISCRETE LOGARITHMS ON A QUANTUM COMPUTER*

PETER W. SHOR†

Abstract. A digital computer is generally believed to be an efficient universal computing device; that is, it is believed able to simulate any physical computing device with an increase in computation time by at most a polynomial factor. This may not be true when quantum mechanics is taken into consideration. This paper considers factoring integers and finding discrete logarithms, two problems which are generally thought to be hard on a classical computer and which have been used as the basis

Multiplication on quantum computers

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... **with as few gates and qubits as possible.**

Multiplication on quantum computers

Today's goal: implement the following unitaries

$$\mathcal{U}_{q\times q} |x\rangle |y\rangle |w\rangle = |x\rangle |y\rangle |w + xy\rangle$$

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Overview

1. Fast multiplication (few gates)

Overview

1. Fast multiplication (few gates)
2. Quantum multiplication (few qubits)

Overview

1. Fast multiplication (few gates)
2. Quantum multiplication (few qubits)
3. Fast quantum multiplication (few gates + qubits)

Background: schoolbook multiplication

The "schoolbook" method: $xy = \sum_{ij}(2^i x_i)(2^j y_j) = \sum_{ij} 2^{i+j} x_i y_j$

					1	1	0	1
				$\times$	1	0	1	0
					1	0	1	0
			1	0	1	0		
+	1	0	1	0				
1	0	0	0	0	0	1	0	

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Running time: $\mathcal{O}(n^2)$ operations

Background: schoolbook multiplication

Given two n-bit numbers x and y, what if we use base $b = 2^{n/2}$?

$$
\begin{array}{cccc}
 & & x_1 & x_0 \\
\times & & y_1 & y_0 \\
\hline
 & & & x_0 y_0 \\
 & & x_1 y_0 & \\
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$$xy = x_1 y_1 b^2 + x_0 y_1 b + x_1 y_0 b + x_0 y_0$$

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$$xy = x_1 y_1 b^2 + x_0 y_1 b + x_1 y_0 b + x_0 y_0$$

Time remains $\mathcal{O}(n^2)$, because $4(n/2)^2 = n^2$

Background: Karatsuba multiplication

$$xy = x_1y_1b^2 + (x_0y_1 + x_1y_0)b + x_0y_0$$

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$$xy = x_1 y_1 b^2 + (x_0 y_1 + x_1 y_0) b + x_0 y_0$$

Observation: $x_0 y_1 + x_1 y_0 = (x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0$

Can compute xy with only three multiplications of size $n/2$:

1. $x_1 y_1$
2. $x_0 y_0$
3. $(x_1 + x_0)(y_1 + y_0)$

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Computational cost: $3(n/2)^2 = \frac{3}{4} n^2 = \mathcal{O}(n^2)$

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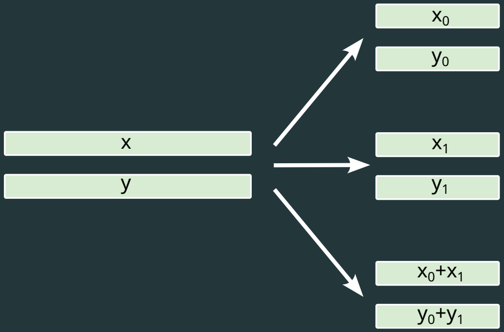

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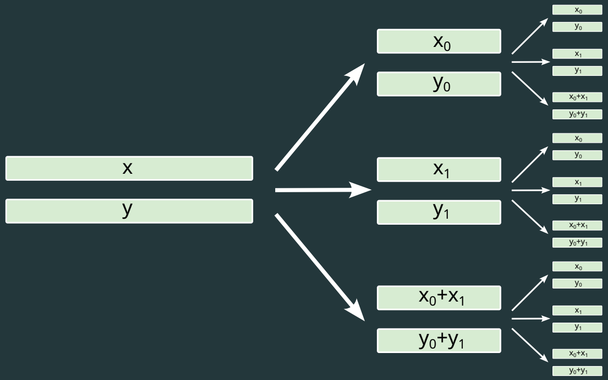

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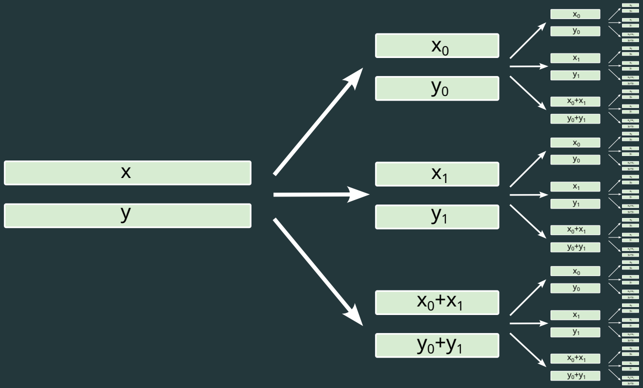

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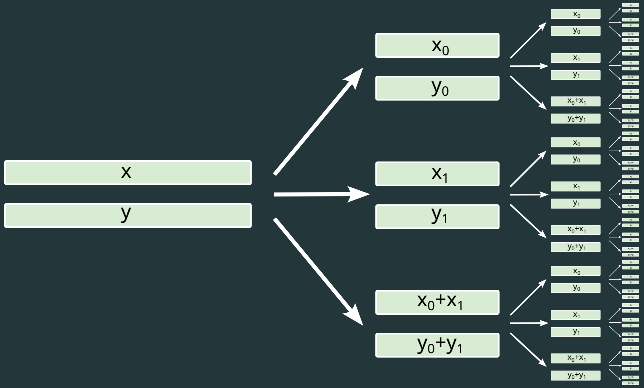
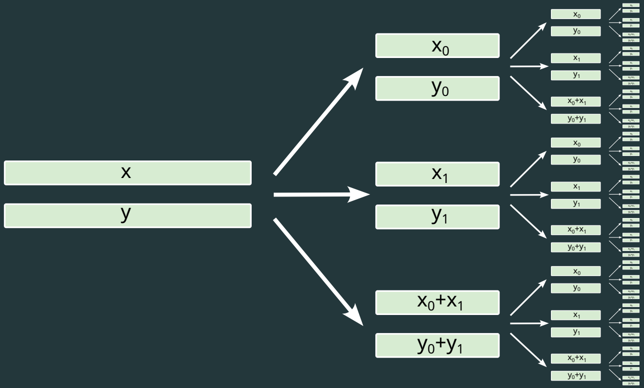


Depth: $d = \log_2 n$

Background: Karatsuba multiplication



Depth: $d = \log_2 n$

Operations: 3^d

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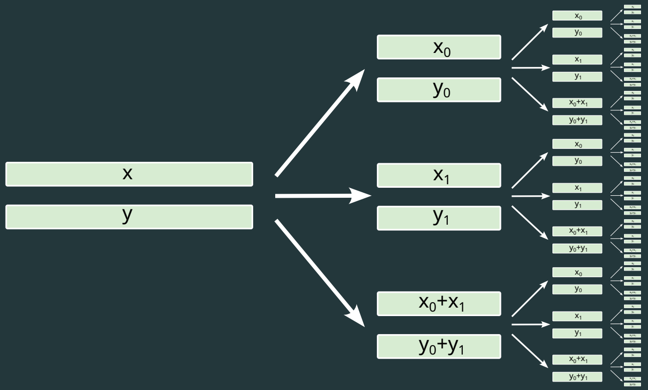

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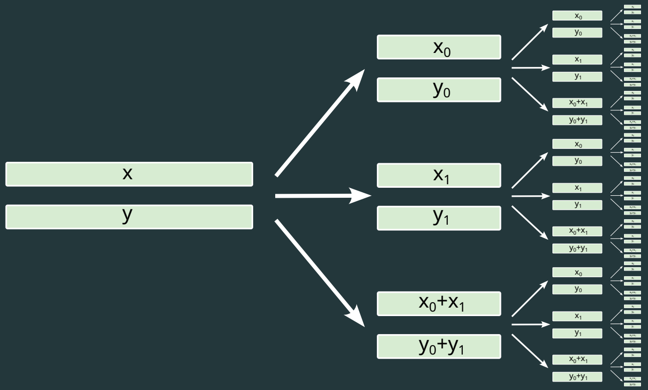
Recursion relation: $T(n) \sim 3T(n/2)$

Background: Karatsuba multiplication



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Operations: 3^d

Recursion relation: $T(n) \sim 3T(n/2)$

Solution: $T(n) = \mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58\cdots})$

Background: Karatsuba multiplication

Question: why don't we always do this, classically?

Answer: the extra complexity isn't always worth it!

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GNU multiple-precision arithmetic library cutoff: 2176 bit numbers

Background: Toom-Cook multiplication

Break the n bit numbers into k chunks of n/k bits.

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Algorithm	Gate count
Schoolbook	$\mathcal{O}(n^2)$
$k=2$	$\mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58\cdots})$
$k=3$	$\mathcal{O}(n^{\log_3 5}) = \mathcal{O}(n^{1.46\cdots})$
$k=4$	$\mathcal{O}(n^{\log_4 7}) = \mathcal{O}(n^{1.40\cdots})$

Summary: fast multiplication

- “Standard” multiplication requires time $\mathcal{O}(n^2)$ operations
- Faster algorithms exist, but have large constant factors

Can these fast circuits be made quantum?

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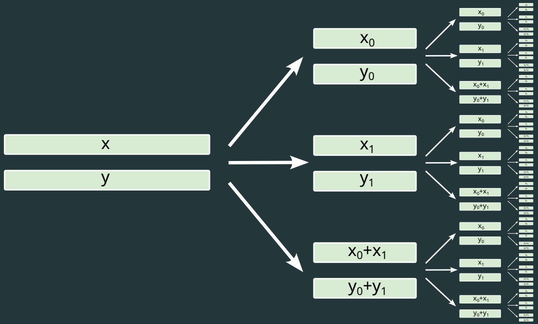
Challenge: making recursive algorithms reversible



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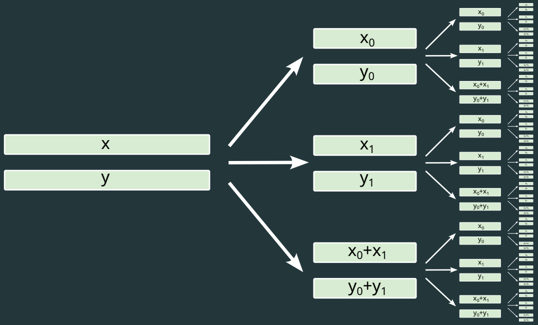

Work	Qubits
Kowada et al. '06	$\mathcal{O}(n^{1.58\cdots})$
Parent et al. '18	$\mathcal{O}(n^{1.43\cdots})$
Gidney '19	$\mathcal{O}(n)$

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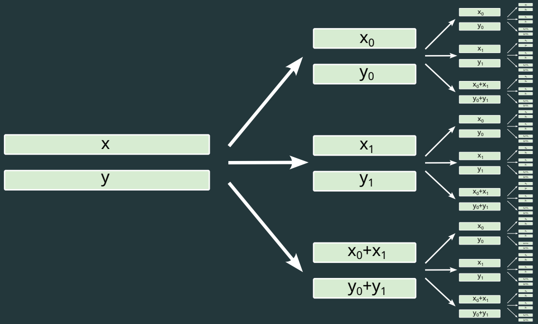

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Gidney ’19 requires over **12,000 ancilla qubits** for 2048-bit multiplication.

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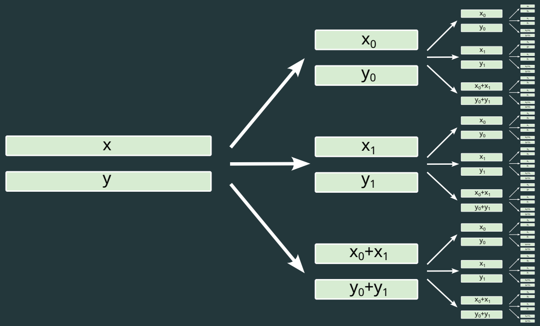

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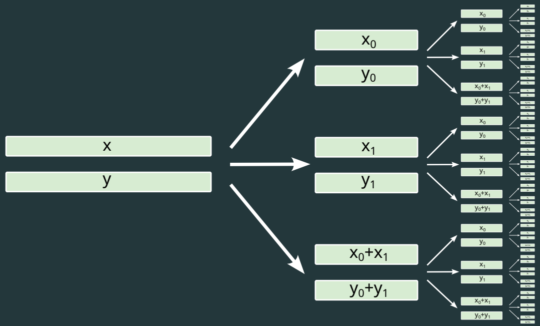

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Result: Fast multiplication using 1 ancilla

A fundamentally quantum way of doing arithmetic

Quantum Fourier transform:

$$QFT\,|w\rangle = \sum_z \exp\left(\frac{2\pi i w z}{2^n}\right)|z\rangle$$

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$$|xy\rangle = \mathrm{QFT}^{-1}\sum_z \exp\left(\frac{2\pi i x y z}{2^n}\right)|z\rangle$$

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1) Generate $|x\rangle\,|y\rangle \sum_z |z\rangle$, 2) apply a phase rotation of $\exp\left(\frac{2\pi i x y z}{2^n}\right)$, 3) apply QFT^{-1}

A fundamentally quantum way of doing arithmetic

How do we apply $\exp\left(\frac{2\pi i x y z}{2^n}\right)$?

A fundamentally quantum way of doing arithmetic

How do we apply $\exp\left(\frac{2\pi ixyz}{2^n}\right)$?

$$xy = \sum_{i,j} 2^i 2^j x_i y_j$$

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How do we apply $\exp\left(\frac{2\pi i xyz}{2^n}\right)$?

$$xyz = \sum_{i,j,k} 2^i 2^j 2^k x_i y_j z_k$$

$$\exp\left(\frac{2\pi i xyz}{2^n}\right) = \prod_{i,j,k} \exp\left(\frac{2\pi i 2^{i+j+k}}{2^n} x_i y_j z_k\right)$$

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x_i, y_j, z_k are binary values—apply phase only if they all are equal to 1!

A series of CCR_ϕ gates between the bits of $|x\rangle$, $|y\rangle$, and $|z\rangle$!

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$$\exp\left(\frac{2\pi i x y z}{2^n}\right) = \prod_{i,j,k} \exp\left(\frac{2\pi i 2^{i+j+k}}{2^n} x_i y_j z_k\right)$$

The downside:

A fundamentally quantum way of doing arithmetic

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$$\exp\left(\frac{2\pi i a x z}{2^n}\right) = \prod_{i,j} \exp\left(\frac{2\pi i a 2^{i+j}}{2^n} x_i z_j\right)$$

Here: $\mathcal{O}(n^2)$ controlled phase rotations (matches Schoolbook algorithm)

Fast quantum multiplication

Main question: Can we combine fast multiplication with Fourier arithmetic to get the benefits of both?

Fast classical-quantum multiplication

Goal: $\mathcal{U}(a) |x\rangle |0\rangle = |x\rangle |ax\rangle$

Fast classical-quantum multiplication

Goal: Apply phase $\exp\left(\frac{2\pi i a}{2^n}xz\right)$; x and z are quantum

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Goal: Implement $\text{PhaseProduct}(\phi)\,|x\rangle\,|z\rangle = \exp\left(i\phi xz\right)|x\rangle\,|z\rangle$

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Previously:

$$\exp(i\phi xz) = \prod_{i,j} \exp\left(i\phi 2^{i+j} x_i z_j\right)$$

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Karatsuba:

$$xz = 2^n x_1 z_1 + 2^{n/2}((x_0 + x_1)(z_0 + z_1) - x_0 z_0 - x_1 z_1) + x_0 z_0$$

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Plugging in Karatsuba:

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How are we supposed to reuse values in the *phase*?

Fast classical-quantum multiplication

Goal: Implement PhaseProduct(ϕ) $|x\rangle |z\rangle = \exp(i\phi xz) |x\rangle |z\rangle$

We want to split the phase ϕxz into the sum of many phases, which are easy to implement.

Karatsuba:

$$xz = 2^n x_1 z_1 + 2^{n/2}((x_0 + x_1)(z_0 + z_1) - x_0 z_0 - x_1 z_1) + x_0 z_0$$

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Re-ordering Karatsuba:

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Plugging in reordered Karatsuba:

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Each of these has the same structure, but on half as many qubits $\rightarrow$ do it recursively!

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Complexity: $T(n) = 3T(n/2) \Rightarrow \mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58\cdots})$ gates!

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Fast quantum-quantum multiplication

Goal: $\mathcal{U} |x\rangle |y\rangle |0\rangle = |x\rangle |y\rangle |xy\rangle$

Fast quantum-quantum multiplication

Goal: Apply phase $\exp\left(\frac{2\pi i}{2^n}xyz\right)$; x, y, and z are quantum

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(n^3 doubly-controlled phase rotations)

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Doesn't work in the phase!!

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k	Runtime
2	$\mathcal{O}(n^2)$
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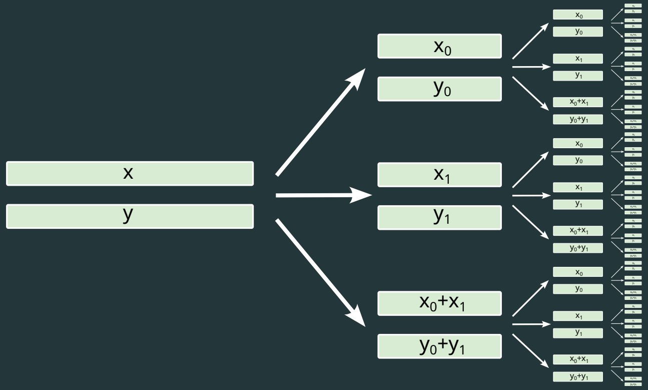
These runtimes are achieved with 2 ancilla qubits.

Depth considerations

Parallelization is natural—achieving depth $\mathcal{O}(n)$ is easy!



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Claim: Can implement PhaseProduct and PhaseTripleProduct in sub-linear time, using $\mathcal{O}(n)$ ancillas!

Suprisingly, multiplication itself is bottlenecked by QFT

Modular arithmetic

So far: have been using phase

$$\exp\left(2\pi i \frac{xyz}{2^n}\right)$$

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Output register requires $n + \mathcal{O}(\log(1/\epsilon))$ qubits

Summary

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1 ancilla qubit

k	Gates
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2 ancilla qubits

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Exact QFT in $\mathcal{O}(n^{1.46})$ gates using 1 ancilla

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- **How well can we optimize explicit circuits?**

Thank you!

Greg Kahanamoku-Meyer — gkm@berkeley.edu

Backup

Fast classical-quantum multiplication: algorithm

PhaseProduct($\phi, |x\rangle, |z\rangle$)

Input: Quantum state $|x\rangle |z\rangle$, classical value ϕ

Output: Quantum state $\exp(i\phi xz) |x\rangle |z\rangle$

1. Split $|x\rangle$ and $|z\rangle$ in half, as $|x_1\rangle |x_0\rangle$ and $|z_1\rangle |z_0\rangle$
2. Apply PhaseProduct($(2^n - 2^{n/2})\phi, |x_1\rangle, |z_1\rangle$)
3. Apply PhaseProduct($(1 - 2^{n/2})\phi, |x_0\rangle, |z_0\rangle$)
4. Add $|x_1\rangle$ to $|x_0\rangle$, and $|z_1\rangle$ to $|z_0\rangle$. Registers now hold $|x_1\rangle |x_0 + x_1\rangle |z_1\rangle |z_0 + z_1\rangle$.
5. Apply PhaseProduct($2^{n/2}\phi, |x_0 + x_1\rangle, |z_0 + z_1\rangle$).
6. Subtract $|x_1\rangle$, $|z_1\rangle$ to return to registers to $|x_1\rangle |x_0\rangle |z_1\rangle |z_0\rangle$.