Fast quantum integer multiplication with very few ancillas

Gregory D. Kahanamoku-Meyer, Norman Y. Yao

October 19, 2023

Motivation

Arithmetic on quantum computers: why do we care?

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Classically verifiable quantum advantage from a computational Bell test

Gregory D. Kahanamoku-Meyer[1], Soonwon Choi[1], Umesh V. Vazirani[2] and Norman Y. Yao[1]

Existing experimental demonstrations of quantum computational advantage have had the limitation that verifying the correctness of the quantum device requires exponentially costly classical computations. Here we propose and analyse an interactive protocol for demonstrating quantum computational advantage, which is efficiently classically verifiable. Our protocol relies on a class of cryptographic tools called trapdoor claw-free functions. Although this type of function has been applied to quantum advantage protocols before, our protocol employs a surprising connection to Bell's inequality to avoid the need for a demanding cryptographic property called the adaptive hardcore bit, while maintaining essentially no increase in the quantum circuit complexity and no extra assumptions. Leveraging the relaxed cryptographic requirements of the protocol, we present two trapdoor claw-free function constructions, based on Rabin's function and the Diffie-Hellman problem, which have not been used in this context before. We also

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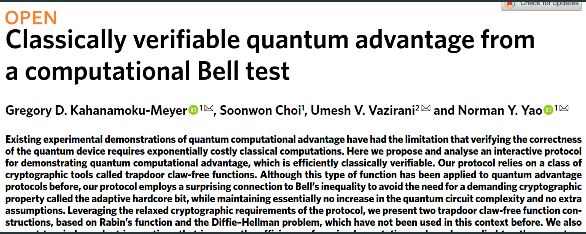
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$x^2 \bmod N$ with this result: $\gtrsim 10^6$ gates, 2000 qubits

Motivation

Arithmetic on quantum computers: why do we care?

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A Cryptographic Test of Quantumness and Certifiable Randomness from a Single Quantum Device

ZVIKA BRAKERSKI, Weizmann Institute of Science, Israel
PAUL CHRISTIANO, OpenAI, USA
URMILA MAHADEV, California Institute of Technology, USA
UMESH VAZIRANI, UC Berkeley, USA
THOMAS VIDICK, California Institute of Technology, USA

We consider a new model for the testing of untrusted quantum devices, consisting of a single polynomial time bounded quantum device interacting with a classical polynomial time verifier. In this model, we propose solutions to two tasks—a protocol for efficient classical verification that the untrusted device is "truly quantum" and a protocol for producing certifiable randomness from a single untrusted quantum device. Our solution relies on the existence of a new cryptographic primitive for constraining the power

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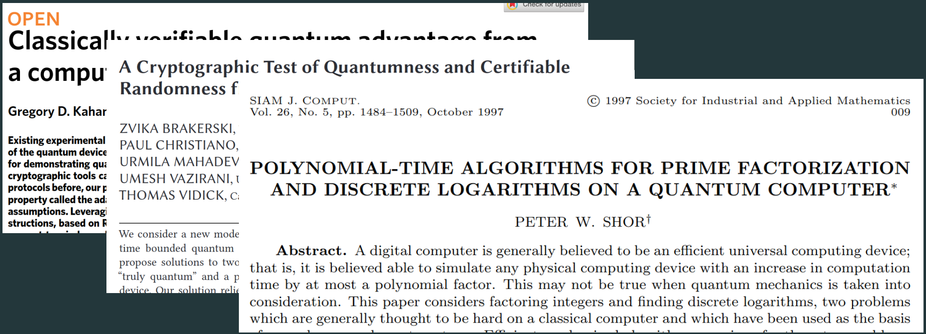

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Vol. 26, No. 5, pp. 1484–1509, October 1997 009

POLYNOMIAL-TIME ALGORITHMS FOR PRIME FACTORIZATION AND DISCRETE LOGARITHMS ON A QUANTUM COMPUTER*

PETER W. SHOR†

Abstract. A digital computer is generally believed to be an efficient universal computing device; that is, it is believed able to simulate any physical computing device with an increase in computation time by at most a polynomial factor. This may not be true when quantum mechanics is taken into consideration. This paper considers factoring integers and finding discrete logarithms, two problems which are generally thought to be hard on a classical computer and which have been used as the basis

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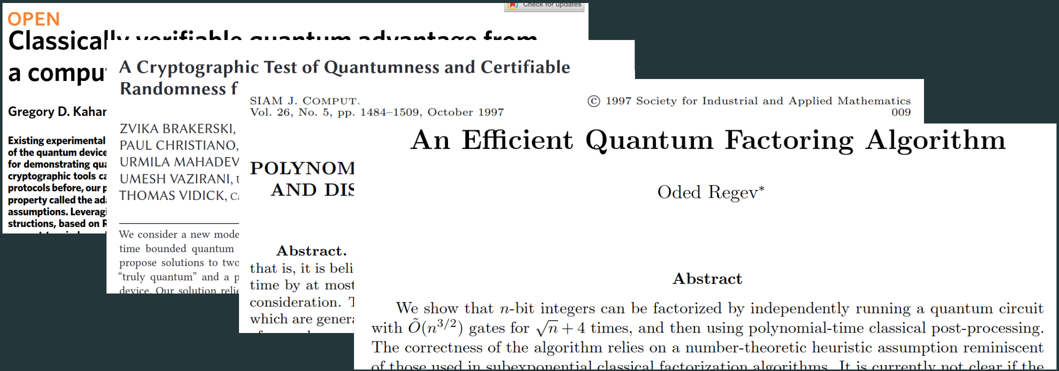

Multiplication on quantum computers

Today's goal: implement the following unitaries

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... with as few gates and qubits as possible.

Multiplication on quantum computers

Today's goal: implement the following unitaries

$$\mathcal{U}_{q\times q} \left|x\right\rangle \left|y\right\rangle \left|w\right\rangle = \left|x\right\rangle \left|y\right\rangle \left|w + xy\right\rangle$$

$$\mathcal{U}_{c\times q}(a) \left|x\right\rangle \left|w\right\rangle = \left|x\right\rangle \left|w + ax\right\rangle$$

... **with as few gates and qubits as possible.**

Background: schoolbook multiplication

The “schoolbook” method: $xy = \sum_{ij}(2^i x_i)(2^j y_j) = \sum_{ij} 2^{i+j} x_i y_j$

				1	1	0	1
			$\times$	1	0	1	0
				1	0	1	0
		1	0	1	0		
+	1	0	1	0			
1	0	0	0	0	0	1	0

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$$
\begin{array}{cccccccc}
 & & & & 1 & 1 & 0 & 1 \\
 & & & \times & 1 & 0 & 1 & 0 \\
\hline
 & & & & 1 & 0 & 1 & 0 \\
 & & 1 & 0 & 1 & 0 & & \\
+ & 1 & 0 & 1 & 0 & & & \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

Running time: $\mathcal{O}(n^2)$ operations

Background: schoolbook multiplication

Given two n-bit numbers x and y, what if we use base $b = 2^{n/2}$?

$$
\begin{array}{rcccc}
 & & x_1 & x_0 \\
\times & & y_1 & y_0 \\
\hline
 & & & x_0 y_0 \\
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$$xy = x_1 y_1 b^2 + x_0 y_1 b + x_1 y_0 b + x_0 y_0$$

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$$xy = x_1 y_1 b^2 + x_0 y_1 b + x_1 y_0 b + x_0 y_0$$

Time remains $\mathcal{O}(n^2)$, because $4(n/2)^2 = n^2$

Background: Karatsuba multiplication

$$xy = x_1 y_1 b^2 + (x_0 y_1 + x_1 y_0) b + x_0 y_0$$

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$$xy = x_1 y_1 b^2 + (x_0 y_1 + x_1 y_0) b + x_0 y_0$$

Observation: $x_0 y_1 + x_1 y_0 = (x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0$

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Can compute xy with only three multiplications of size $\log b = n/2$:

1. $x_1 y_1$
2. $x_0 y_0$
3. $(x_1 + x_0)(y_1 + y_0)$

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Can compute xy with only three multiplications of size $\log b = n/2$:

1. x_1y_1
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Computational cost: $3(n/2)^2 = \frac{3}{4}n^2 = \mathcal{O}(n^2)$

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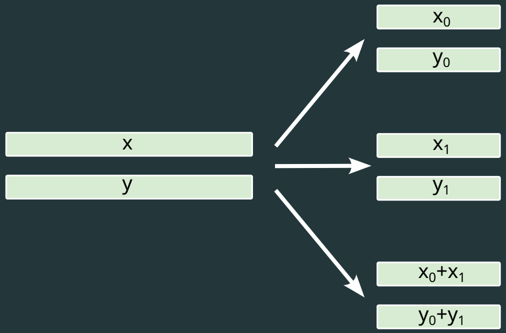

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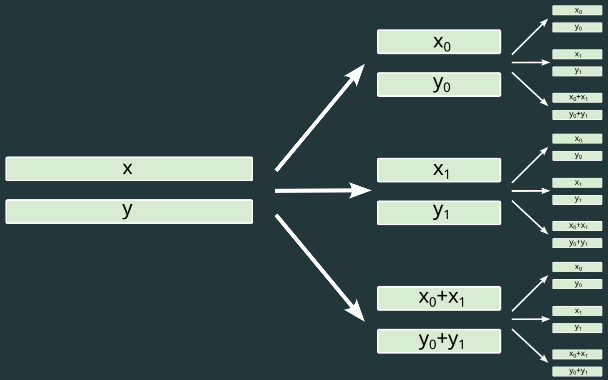

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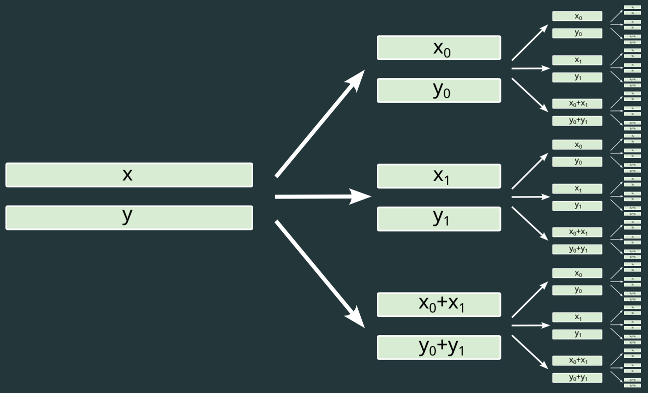

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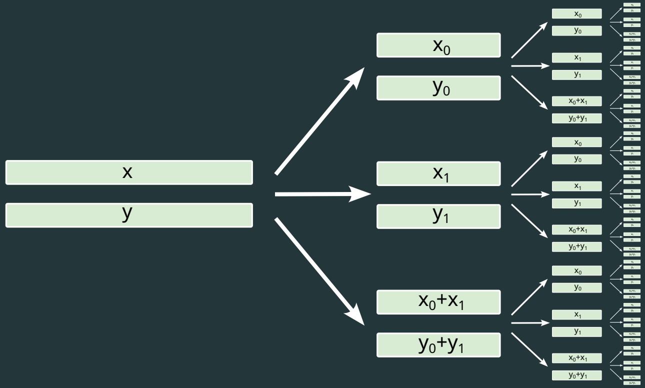
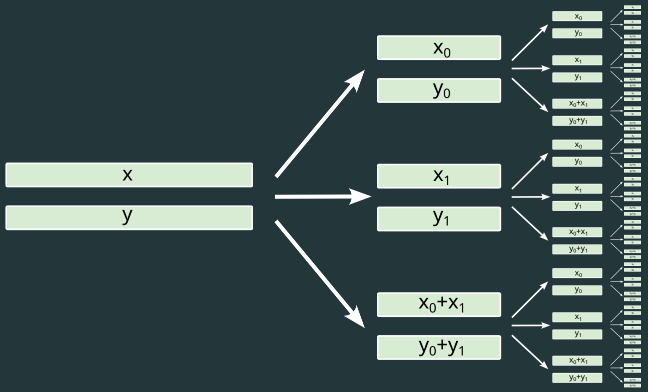


Depth: $d = \log_2 n$

Background: Karatsuba multiplication



Depth: $d = \log_2 n$

Operations: 3^d

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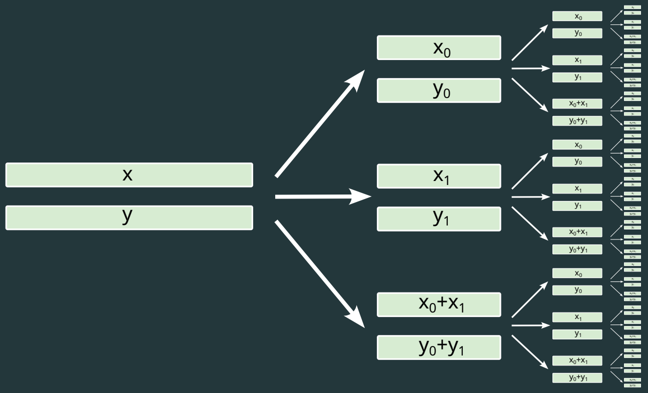

Depth: $d = \log_2 n$

Operations: 3^d

Cost: $\mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58\cdots})$

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Question: why don't we always do this, classically?

Answer: the extra complexity isn't always worth it!

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GNU multiple-precision arithmetic library cutoff: 2176 bit numbers

Can these fast circuits be made quantum?

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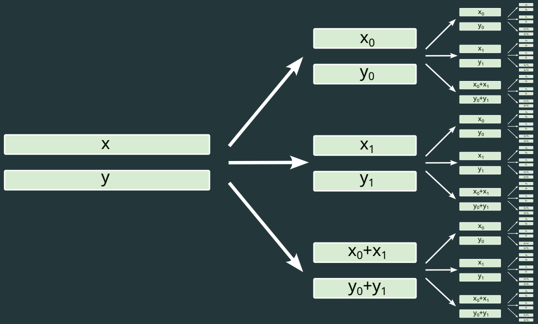
Challenge: making recursive algorithms reversible



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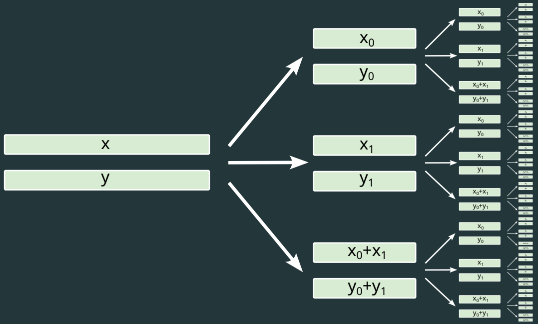

Work	Qubits
Kowada et al. '06	$\mathcal{O}(n^{1.58\cdots})$
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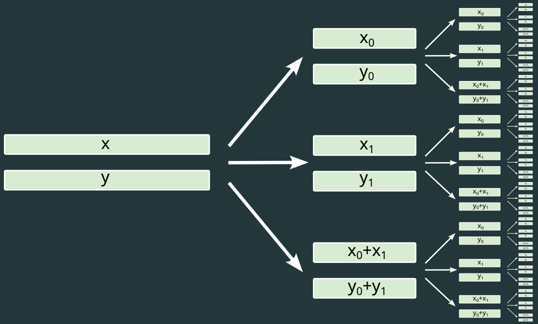

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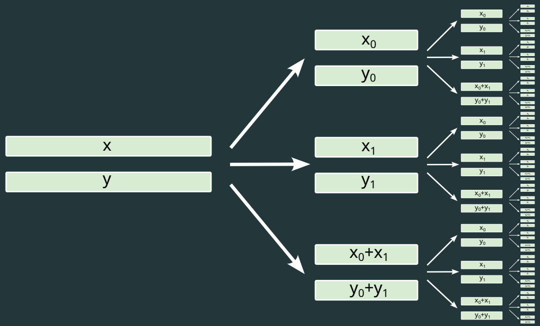

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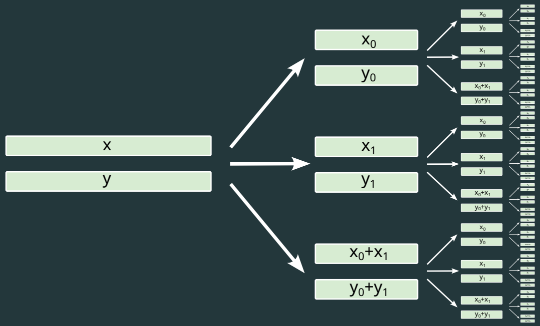

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Result: Fast multiplication using 1 ancilla

A fundamentally quantum way of doing arithmetic

[Draper '04]: Arithmetic in Fourier space

$$|xy\rangle = \mathrm{QFT}^{-1} \sum_z \exp\left(\frac{2\pi i xyz}{2^n}\right) |z\rangle$$

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$$\exp\left(\frac{2\pi i xyz}{2^n}\right) = \prod_{i,j,k} \exp\left(\frac{2\pi i 2^{i+j+k}}{2^n} x_i y_j z_k\right)$$

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x_i, y_j, z_k are binary values—apply phase only if they all are equal to 1!

A series of CCR_ϕ gates between the bits of $|x\rangle$, $|y\rangle$, and $|z\rangle$!

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$$\exp\left(\frac{2\pi i xyz}{2^n}\right) = \prod_{i,j,k} \exp\left(\frac{2\pi i 2^{i+j+k}}{2^n} x_i y_j z_k\right)$$

The downside:

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$$\exp\left(\frac{2\pi i axz}{2^n}\right) = \prod_{i,j} \exp\left(\frac{2\pi i a 2^{i+j}}{2^n} x_i z_j\right)$$

Here: $\mathcal{O}(n^2)$ controlled phase rotations (matches Schoolbook algorithm)

Fast quantum multiplication

Main question: Can we combine fast multiplication with Fourier arithmetic to get the benefits of both?

Fast classical-quantum multiplication

Goal: $\mathcal{U}(a)\,|x\rangle\,|0\rangle = |x\rangle\,|ax\rangle$

Fast classical-quantum multiplication

Goal: Apply phase $\exp\left(\frac{2\pi i a}{2^n} xz\right)$; x and z are quantum

Fast classical-quantum multiplication

Goal: Implement $\text{PhaseProduct}(\phi)\,|x\rangle\,|z\rangle = \exp(i\phi xz)\,|x\rangle\,|z\rangle$

Fast classical-quantum multiplication

Goal: Implement PhaseProduct(ϕ) $|x\rangle\,|z\rangle = \exp\left(i\phi xz\right)|x\rangle\,|z\rangle$

We want to split the phase ϕxz into the sum of many phases, which are easy to implement.

Fast classical-quantum multiplication

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We want to split the phase ϕxz into the sum of many phases, which are easy to implement.

Previously:

$$\exp(i\phi xz) = \prod_{i,j} \exp\left(i\phi 2^{i+j} x_i z_j\right)$$

Fast classical-quantum multiplication

Goal: Implement $\text{PhaseProduct}(\phi)\left|x\right\rangle\left|z\right\rangle = \exp\left(i\phi xz\right)\left|x\right\rangle\left|z\right\rangle$

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Karatsuba:

$$xz = 2^n x_1 z_1 + 2^{n/2}((x_0 + x_1)(z_0 + z_1) - x_0 z_0 - x_1 z_1) + x_0 z_0$$

Fast classical-quantum multiplication

Goal: Implement PhaseProduct(ϕ) $|x\rangle\,|z\rangle = \exp(i\phi xz)\,|x\rangle\,|z\rangle$

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Plugging in Karatsuba:

$$\begin{aligned}\exp(i\phi xz) = &\exp(i\phi 2^n x_1 z_1)\\ &\cdot \exp(i\phi x_0 z_0)\\ &\cdot \exp\left(i\phi 2^{n/2}((x_0+x_1)(z_0+z_1) - x_0 z_0 - x_1 z_1)\right)\end{aligned}$$

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How are we supposed to reuse values in the *phase*?

Fast classical-quantum multiplication

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Re-ordering Karatsuba:

$$xz = (2^n - 2^{n/2})x_1 z_1 + 2^{n/2}(x_0 + x_1)(z_0 + z_1) + (1 - 2^{n/2})x_0 z_0$$

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Plugging in reordered Karatsuba:

$$\begin{aligned}\exp(i\phi xz) = &\exp\left(i\phi(2^n - 2^{n/2})x_1 z_1\right)\\ &\cdot \exp\left(i\phi(1 - 2^{n/2})x_0 z_0\right)\\ &\cdot \exp\left(i\phi 2^{n/2}(x_0 + x_1)(z_0 + z_1)\right)\end{aligned}$$

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$$\begin{aligned}\exp\left(i\phi xz\right) = &\exp\left(i\phi_1 x_1 z_1\right)\\ &\cdot\exp\left(i\phi_2 x_0 z_0\right)\\ &\cdot\exp\left(i\phi_3 (x_0 + x_1)(z_0 + z_1)\right)\end{aligned}$$

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Each of these has the same structure, but on half as many qubits $\rightarrow$ do it recursively!

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Recursion relation: $T(n) = 3T(n/2)$

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Recursion relation: $T(n) = 3T(n/2) \Rightarrow\ \mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58\cdots})$ gates!

How many qubits do we need?

Splitting registers $|x\rangle \to |x_1\rangle |x_0\rangle$ and $|z\rangle \to |z_1\rangle |z_0\rangle$, can immediately do

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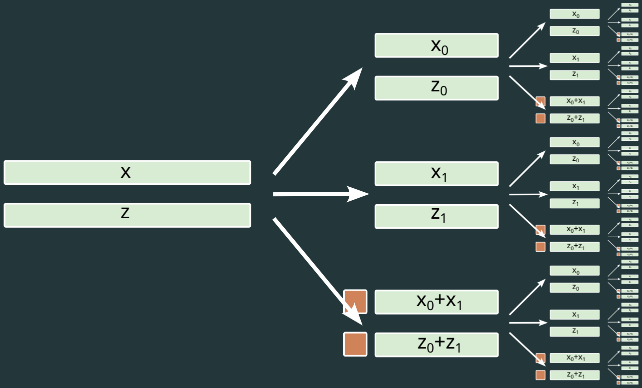

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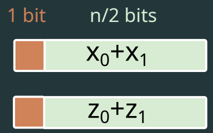
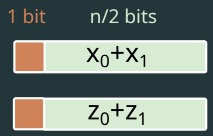


How many qubits do we need?

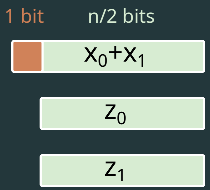
Total number of ancillas: 2



Idea: "Shave off" the high bit before recursing

How many qubits do we need?

Total number of ancillas: 1



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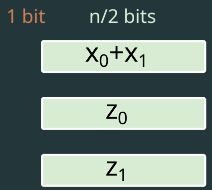

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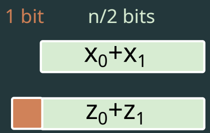

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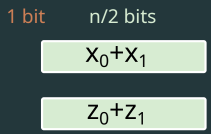

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Making it go faster

So far: $\mathcal{O}(n^{1.58})$ gates using 1 ancilla

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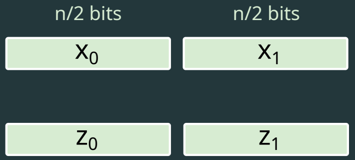
Can we make it go faster?

Background: Toom-Cook multiplication

Let $b = 2^{n/2}$.

$$x = x_1 b + x_0$$

$$z = z_1 b + z_0$$



Background: Toom-Cook multiplication

Let $b = 2^{n/k}$.

$$x = \sum_{i=0}^{k-1} x_i b^i$$

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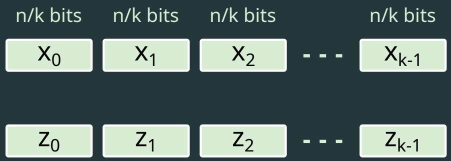
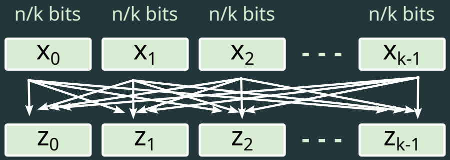


Background: Toom-Cook multiplication

Let $b = 2^{n/k}$.

$$xz = \left(\sum_{i=0}^{k-1} x_i b^i\right)\left(\sum_{i=0}^{k-1} z_i b^i\right)$$



Schoolbook: k^2 multiplications of size n/k

Background: Toom-Cook multiplication

$$x(b) = \sum_{i=0}^{k-1} x_i b^i$$

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- For any point w, $p(w) = x(w)z(w)$

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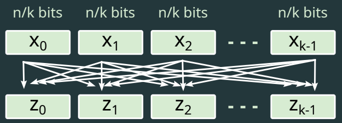
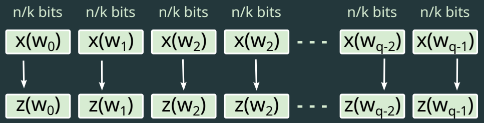
Facts:

- For any point w, $p(w) = x(w)z(w)$
- $p(b)$ has degree $2(k-1) \Rightarrow$ uniquely determined by $q = 2(k-1)+1$ points w_ℓ!

Background: Toom-Cook multiplication

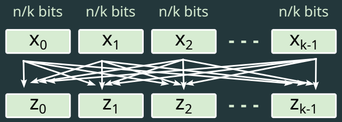
1. Compute $x(w_\ell)$, $z(w_\ell)$ at q points w_ℓ

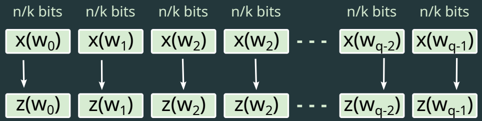




Only $2k - 1$ multiplications of size n/k!

1. Compute $x(w_\ell)$, $z(w_\ell)$ at q points w_ℓ
2. Pointwise multiply





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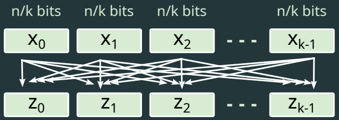

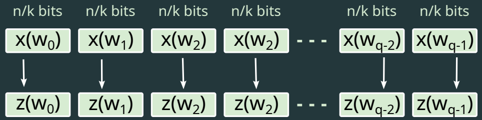

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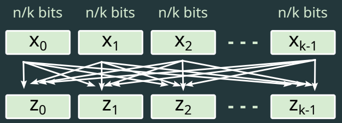

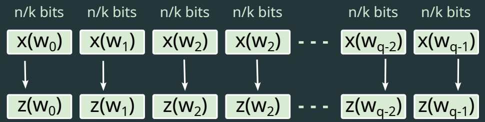

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Karatsuba in the language of Toom-Cook

Karatsuba is Toom-Cook with $k = 2$

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Let $w \in \{0, \infty, 1\}$

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$$p(b) = p(\infty)b^2 + [p(1) - p(\infty) - p(0)]\, b + p(0)$$

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$$p(2^{n/2}) = x_1 z_1 2^n + \left[(x_0 + x_1)(z_0 + z_1) - x_1 z_1 - x_0 z_0\right] 2^{n/2} + x_0 z_0$$

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Complexity vs. k

Toom-Cook has asymptotic complexity $\mathcal{O}(n^{log_k(2k-1)})$

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Algorithm	Gate count
Schoolbook	$\mathcal{O}(n^2)$
$k = 2$	$\mathcal{O}(n^{1.58\cdots})$
$k = 3$	$\mathcal{O}(n^{1.46\cdots})$
$k = 4$	$\mathcal{O}(n^{1.40\cdots})$
$\vdots$	$\vdots$

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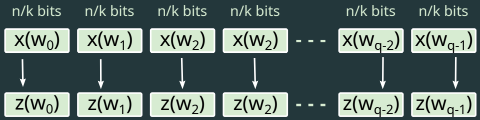
These are the gate counts for our classical-quantum multiplication!

Overhead moves to classical precomputation

1. Compute $x(w_\ell), z(w_\ell)$ at q points w_ℓ
2. Pointwise multiply
3. Interpolate $p(b)$
4. Evaluate $p(2^{n/k})$



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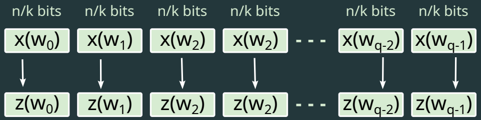

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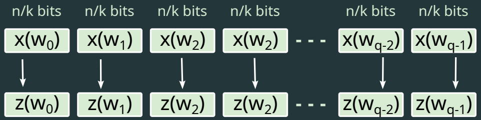

$$\phi xz = \sum_{\ell=0}^{2k-2} \phi_\ell x(w_\ell) z(w_\ell) \tag{1}$$

Much of the overhead has moved to classical precomputation!

Fast quantum-quantum multiplication

Goal: $\mathcal{U} |x\rangle |y\rangle |0\rangle = |x\rangle |y\rangle |xy\rangle$

Fast quantum-quantum multiplication

Goal: Apply phase $\exp\left(\frac{2\pi i}{2^n}xyz\right)$; x, y, and z are quantum

Fast quantum-quantum multiplication

Goal: Implement $\text{PhaseTripleProduct}(\phi) \left|x\right\rangle \left|y\right\rangle \left|z\right\rangle = \exp\left(i\phi xyz\right) \left|x\right\rangle \left|y\right\rangle \left|z\right\rangle$

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Doesn't work in the phase!!

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$$p(b) = x(b)y(b)z(b)$$

$p(b)$ has degree $q = 3k - 2$

Example: Generalizing Karatsuba's method

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$$
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xyz = & (2^{3n/2} - 2^{n/2}) x_1 y_1 z_1 \\
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k	Gates $\mathcal{O}(n^{\log_k(3k-2)})$
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2	$\mathcal{O}(n^2)$
3	$\mathcal{O}(n^{1.77\cdots})$
4	$\mathcal{O}(n^{1.66\cdots})$
5	$\mathcal{O}(n^{1.59\cdots})$
6	$\mathcal{O}(n^{1.55\cdots})$
$\vdots$	$\vdots$

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For any $m < n$, we may implement QFT_{2^n}:

1. Apply QFT_{2^m} on first m qubits
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   - $|x\rangle$ is value of first m qubits
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3. Apply $\mathrm{QFT}_{2^{n-m}}$ on final $n-m$ qubits

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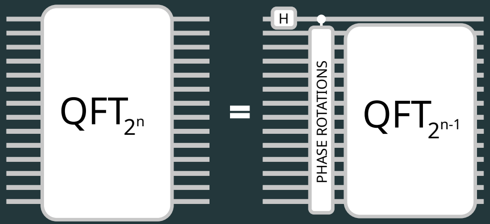

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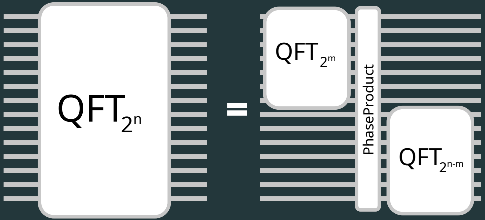

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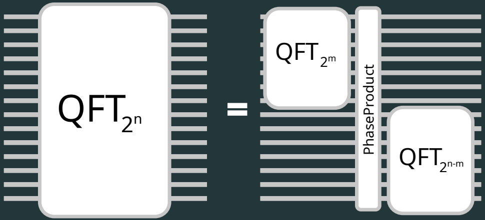
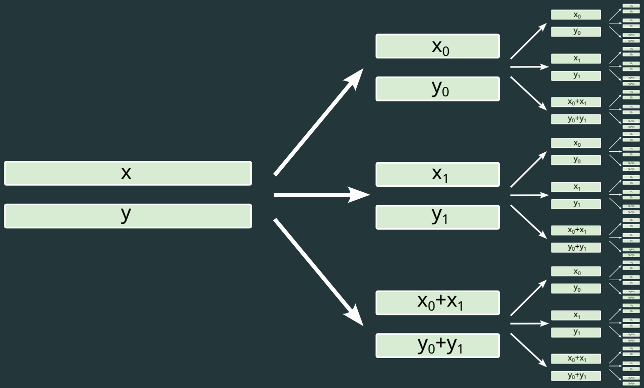


Immediately gives us sub-quadratic exact QFT using only 1 ancilla.

Depth considerations

Parallelization is natural.



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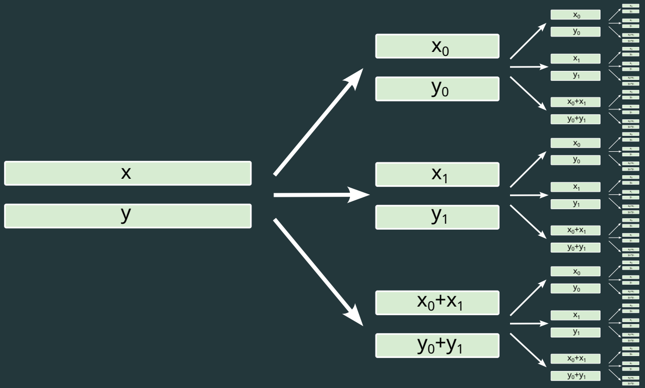
Parallelization is natural.

We have k sub-registers to work with—can do k sub-products in parallel.



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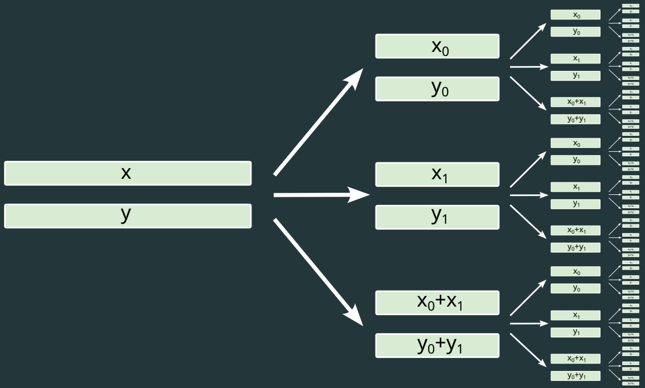

Depth: PhaseProduct in $\mathcal{O}(n^{\log_k 2})$ and PhaseTripleProduct in $\mathcal{O}(n^{\log_k 3})$ using a few more ancillas

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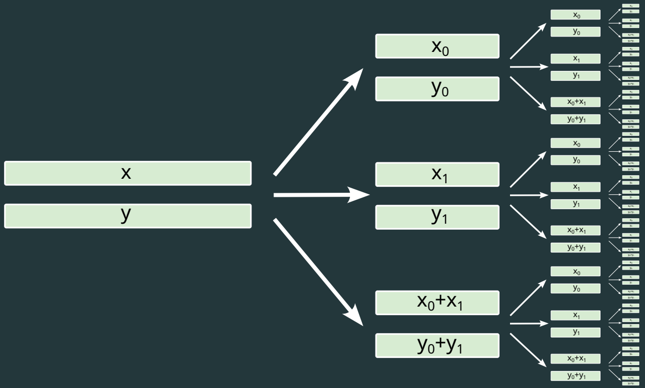

Challenge for multiply: How to do the QFT in sublinear depth with even $\mathcal{O}(n)$ ancillas?

So far: have been using phase

$$\exp\left(2\pi i \frac{xyz}{2^n}\right)$$

Modular arithmetic

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Modular arithmetic

Goal: only use n bits for output modulo N

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Output register requires $n + \mathcal{O}(\log(1/\epsilon))$ qubits

Application: Shor's algorithm

For Shor's algorithm: $\mathcal{O}(n)$ modular classical-quantum multiplications

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Using phase modulo and $k = 4$ multiplier:

Gates: $\mathcal{O}(n^{2.4})$
Total qubits: $2n + \mathcal{O}(\log(n/\epsilon))$

(Here ϵ is error across the whole algorithm)

Application: Shor's algorithm

Cost estimates for one 2048-bit classical-quantum multiplication: (here not modular)

Algorithm	Complexity	Gate count (millions)			Ancilla qubits
		Toffoli	CR_ϕ	Other	
This work	$\mathcal{O}(n^{1.4})$	**0.6**	**0.9**	**2.1**	**50**
Karatsuba [1]	$\mathcal{O}(n^{1.58})$	5.6	—	34	12730
Windowed [1]	$\mathcal{O}(n^2)$	1.8	—	2.5	4106
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(Note: $\sim$ 15% of the CR_ϕ come from approximate QFTs with $\epsilon = 10^{-12}$)

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Open q.: Can we use windowing with our construction?

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Protocol for a "proof of quantumness" requires evaluating $f(x) = x^2 \mod N$

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Cost estimates for protocol with 1024-bit N:

Algorithm	Gate count (millions)			Total qubits
	Toffoli	C^*R_ϕ	Other	
Gate optimized	**0.7**	**0.9**	**0.7**	**2400**
Balanced	**0.9**	**1.0**	**0.9**	**2070**
Qubit optimized	**2.2**	**2.0**	**2.2**	**1560**
"Digital" Karatsuba [2]	1.6	—	1.6	6801
"Digital" Schoolbook [2]	3.5	—	2.9	4097
Prev. Fourier 1 [2]	—	539	—	1025
Prev. Fourier 2 [2]	—	35	—	2062

[2] GDKM, Choi, Vazirani, Yao. "Efficiently-verifiable quantum advantage from a computational Bell test." (arXiv:2104.00687)

Summary

Classical-quantum

1 ancilla qubit

k	Gates
2	$\mathcal{O}(n^{1.58\cdots})$
3	$\mathcal{O}(n^{1.46\cdots})$
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Low crossover—in some cases, already faster for 20 bit inputs!

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- **How well can we optimize explicit circuits (especially the base case)?**

Thank you!

Greg Kahanamoku-Meyer — gkm@berkeley.edu

Backup

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$1024\ CR_\phi \rightarrow 64\ R_\phi$ plus ~ 2048 Toffoli

Fast classical-quantum multiplication: algorithm

PhaseProduct($\phi, |x\rangle, |z\rangle$)

Input: Quantum state $|x\rangle |z\rangle$, classical value ϕ

Output: Quantum state $\exp(i\phi xz) |x\rangle |z\rangle$

1. Split $|x\rangle$ and $|z\rangle$ in half, as $|x_1\rangle |x_0\rangle$ and $|z_1\rangle |z_0\rangle$
2. Apply PhaseProduct($(2^n - 2^{n/2})\phi, |x_1\rangle, |z_1\rangle$)
3. Apply PhaseProduct($(1 - 2^{n/2})\phi, |x_0\rangle, |z_0\rangle$)
4. Add $|x_1\rangle$ to $|x_0\rangle$, and $|z_1\rangle$ to $|z_0\rangle$. Registers now hold $|x_1\rangle |x_0 + x_1\rangle |z_1\rangle |z_0 + z_1\rangle$.
5. Apply PhaseProduct($2^{n/2}\phi, |x_0 + x_1\rangle, |z_0 + z_1\rangle$).
6. Subtract $|x_1\rangle$, $|z_1\rangle$ to return to registers to $|x_1\rangle |x_0\rangle |z_1\rangle |z_0\rangle$.