The Jacobi Factoring Circuit

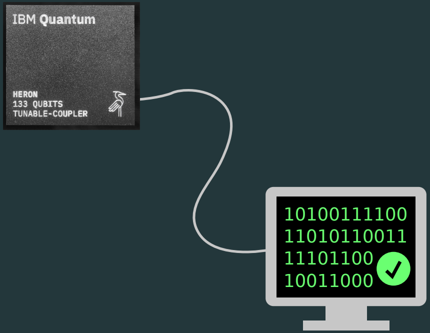
Classically-hard factoring in sublinear quantum space and depth

Gregory D. Kahanamoku-Meyer*, Seyoon Ragavan*, Vinod Vaikuntanathan*, Katherine van Kirk†

*MIT, †Harvard

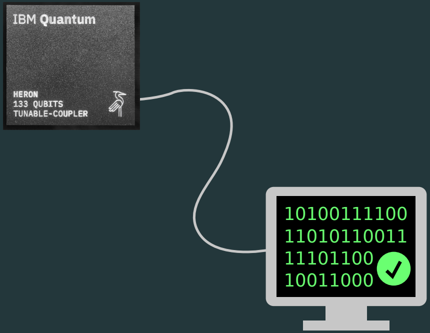
November 19, 2024

Verifiable quantum advantage



How can a single **black-box** device prove its quantum capability to a skeptical **classical** verifier?

Verifiable quantum advantage



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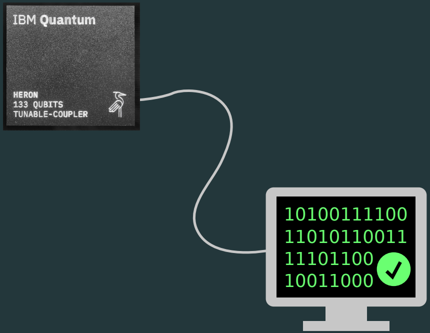

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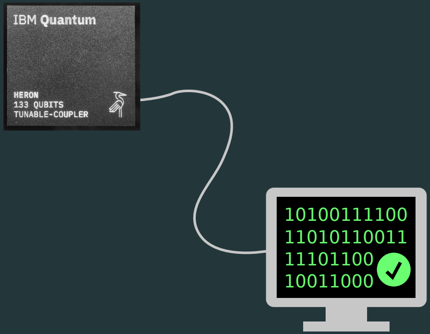

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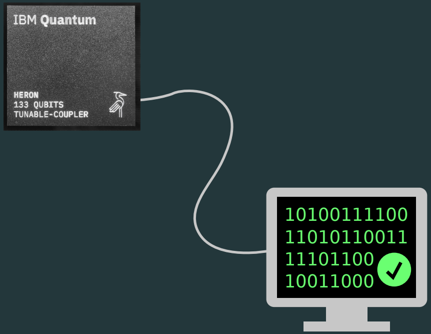

Ideal protocol:

- **Provably classically hard**, reducible to an *established* problem
- **Polynomial-time** classical verification
- **Small circuits** in terms of qubits, gates, and depth

How can a single **black-box** device prove its quantum capability to a skeptical **classical** verifier?

Verifiable quantum advantage via factoring

Algorithms for Quantum Computation:
Discrete Logarithms and Factoring

Peter W. Shor
AT&T Bell Labs
Room 2D-149
600 Mountain Ave.
Murray Hill, NJ 07974, USA

Protocol: Pick primes p, q, ask the quantum device to factor n-bit $N = pq$.

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Gates: $\widetilde{\mathcal{O}}(n^2)$ Depth: $\widetilde{\mathcal{O}}(n)$ Qubits: $\widetilde{\mathcal{O}}(n)$

Verifiable quantum advantage via factoring

An Efficient Quantum Factoring Algorithm

Oded Regev*

Protocol: Pick primes p, q, ask the quantum device to factor n-bit $N = pq$.

Gates: $\widetilde{\mathcal{O}}(n^{3/2})$ **Depth:** $\widetilde{\mathcal{O}}(n^{1/2})$ **Qubits:** $\widetilde{\mathcal{O}}(n)^*$

* with the optimizations of Ragavan and Vaikuntanathan [arXiv:2310.00899]

Verifiable quantum advantage via factoring

Article | Open access | Published: 01 August 2022

Classically verifiable quantum advantage from a computational Bell test

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Protocol:

3-round interactive protocol; quantum device evaluates $x^2 \bmod N$ for n-bit $N = pq$

Gates: $\widetilde{\mathcal{O}}(n)$ **Depth:** $\mathcal{O}(\mathrm{polylog}\, n)$ **Qubits:** $\widetilde{\mathcal{O}}(n)$

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… note it doesn't actually factor the number!

Verifiable quantum advantage via factoring

Algorithm	Gates	Depth	Qubits
Shor	$\widetilde{\mathcal{O}}(n^2)$	$\widetilde{\mathcal{O}}(n)$	$\widetilde{\mathcal{O}}(n)$
Regev + RV23	$\widetilde{\mathcal{O}}(n^{3/2})$	$\widetilde{\mathcal{O}}(n^{1/2})$	$\widetilde{\mathcal{O}}(n)$
$x^2 \bmod N$	$\widetilde{\mathcal{O}}(n)$	$\widetilde{\mathcal{O}}(n^0)$	$\widetilde{\mathcal{O}}(n)$

All algorithms implemented with fast, low-depth multipliers.
Tildes indicate omitted polylog factors.

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"Cool but that's still too many qubits" –every experimentalist when I talk about $x^2 \bmod N$

Verifiable quantum advantage via factoring

For n-bit numbers of the form $N = pq$:

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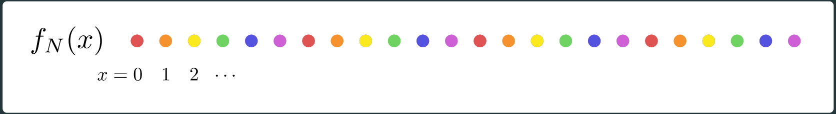
Space and depth proportional to the **length of the factor!**

Why do we need $\mathcal{O}(n)$ qubits?

Find some $f_N(x)$ w/ period *P*, where *P* can be used to find factors



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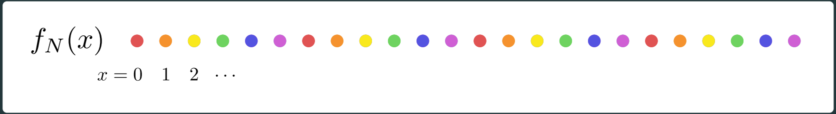
→

Generate

$$\sum_{x=0}^{\text{poly}(P)} |x\rangle\, |f_N(x)\rangle$$



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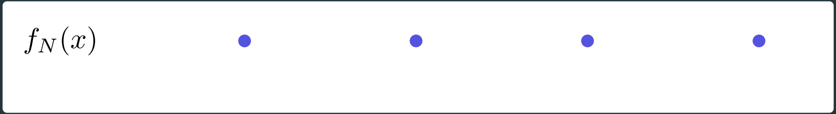
Find some $f_N(x)$ w/ period *P*, where *P* can be used to find factors → **Generate** $\sum_{x=0}^{\text{poly}(P)} |x\rangle |f_N(x)\rangle$ → **Measure, yielding superposition over coset**



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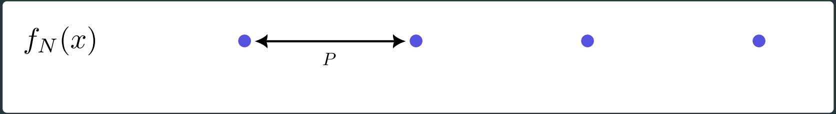

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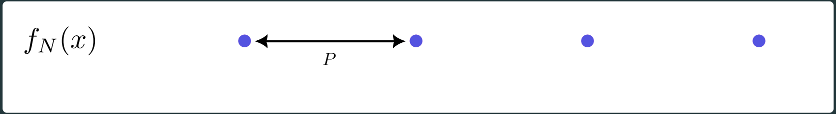

<u>Shor's algorithm:</u> Function: $f_N(x) = a^x \bmod N$ Period: $P = \text{ord}_N(a) = \mathcal{O}(N)$

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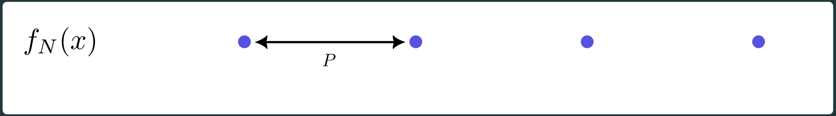

Shor's algorithm: Function: $f_N(x) = a^x \bmod N$ Period: $P = \mathrm{ord}_N(a) = \mathcal{O}(N)$

Could we find a function with smaller period?

Some number theory

Legendre symbol

Legendre symbol

For a prime p:

$$\left(\frac{x}{p}\right) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod{p} \\ 1 & \text{if } \exists\, w \text{ s.t. } w^2 \equiv x \pmod{p} \\ -1 & \text{otherwise} \end{cases}$$

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Legendre symbol is 1) **efficient to compute** given x and p, 2) **periodic** with period p

Jacobi symbol

For a composite number $N = \prod_i p_i$:

$$\left(\frac{x}{N}\right) = \prod_i \left(\frac{x}{p_i}\right)$$

Jacobi symbol is 1) **efficient to compute** given x and N, 2) **periodic** with period...?

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For $N = pq$:

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Period is N—not helpful for factoring!

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For $N = p^2q$:

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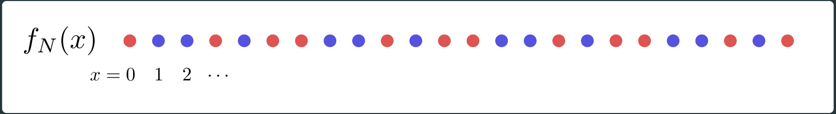
Period is q—exactly what we need!!

Factoring with the Jacobi symbol

Find some $f_N(x)$ w/ period *P*, where *P* can be used to find factors



Jacobi factoring, for $N = p^2 q$:

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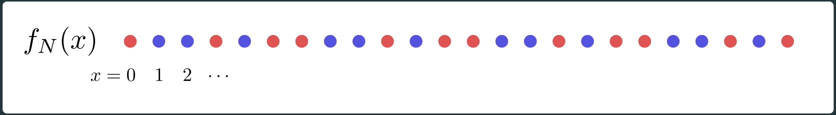

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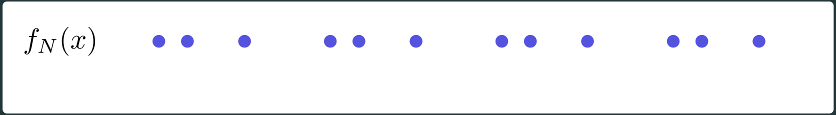

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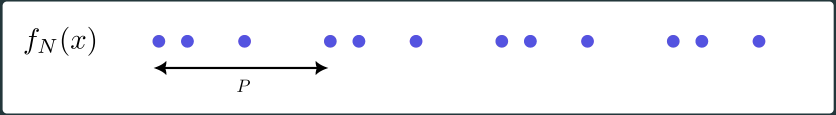

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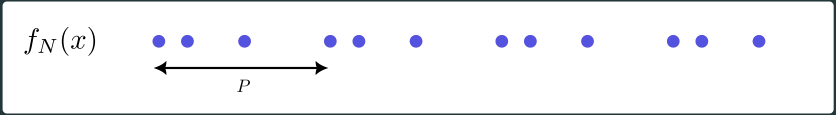

Jacobi factoring, for $N = p^2q$: Function: $f_N(x) = \left(\frac{x}{N}\right)$ Period: $P = q$

Is this going to actually work?

Li, Peng, Du, Suter ’12

LDPS. “An Efficient Exact Quantum Algorithm for the Integer Square-free Decomposition Problem.” Nature Scientific Reports, 2012.

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Their results:

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  - With superposition only to $\mathrm{poly}(Q)$, algorithm still succeeds w.h.p.
- Jacobi symbol can be computed efficiently via standard circuits
  - When quantum input is small, **extremely** efficient quantum circuits exist!

Computing the Jacobi symbol

Goal: Compute $\left(\frac{x}{N}\right)$

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$$\left(\frac{a}{b}\right) \in \{-1, 0, 1\} \tag{1}$$

Computing the Jacobi symbol

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Recall: N is classical, n bits; $|x\rangle$ is quantum, m qubits—and potentially $m \ll n$.

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The "big" input is entirely classical.
Can we implement this circuit using only $O(m)$ qubits?

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Euclidean algorithm

Euclid, Greece, 2000 years ago

Iterate:

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Binary GCD

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Suppose a, b odd

Iterate:

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Both seem to require at least $\mathcal{O}(n)$ qubits, and not reversible...

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find $N' = kx$ s.t. only leading m bits of $N - N'$ are nonzero

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$$\left(\frac{x}{N}\right)$$

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$$\left(\frac{x}{N}\right) \to \left(\frac{N}{x}\right) \to \left(\frac{N - kx}{x}\right) \to \left(\frac{(N - kx)/2^{n-m}}{x}\right)$$

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Overall plan:

$$\left(\frac{x}{N}\right) \to \left(\frac{N}{x}\right) \to \left(\frac{N-kx}{x}\right) \to \left(\frac{(N-kx)/2^{n-m}}{x}\right)$$

Last value has two m-bit inputs; cost is independent of N with standard circuits.

Computing the Jacobi symbol

Idea: For n-bit N and m-bit x,
find $N' = kx$ s.t. only leading m bits of $N - N'$ are nonzero

Overall plan:

$$\left(\frac{x}{N}\right) \to \left(\frac{N}{x}\right) \to \left(\frac{N-kx}{x}\right) \to \left(\frac{(N-kx)/2^{n-m}}{x}\right)$$

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Goal #2: Circuit for $|x\rangle\,|0^m\rangle \to |x\rangle\,|N'\rangle$

Computing the Jacobi symbol

Goal #2: Circuit for $|x\rangle |0^m\rangle \rightarrow |x\rangle |N'\rangle$

```
|x⟩ =                 |1 0 0 1 0 1 1⟩
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Gate count: $\mathcal{O}(nm)$.

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```

Gate count: $\mathcal{O}(nm)$. **We can do better!**

Computing the Jacobi symbol, fast!

Goal #2: Circuit for $|x\rangle\,|0^m\rangle \rightarrow |x\rangle\,|N'\rangle$

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|x⟩ =                                                    |1 0 1 1 1 1 0 1⟩
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```

Computing the Jacobi symbol, fast!

Result: Fast circuit for $|x\rangle\,|0^m\rangle \rightarrow |x\rangle\,|N'\rangle$

Suppose t-bit multiplication costs $G_M(t)$ gates, $D_M(t)$ depth, $S_M(t)$ qubits.

Circuit cost:

Gates: $\mathcal{O}(\frac{n}{m} \cdot G_M(m))$

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Gates: $\mathcal{O}(\frac{n}{m} \cdot G_M(m))$

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Space: $\mathcal{O}(S_M(m))$

Computing the Jacobi symbol

Overall plan:

$$\left(\frac{x}{N}\right) \to \left(\frac{N}{x}\right) \to \left(\frac{N - kx}{x}\right) \to \left(\frac{(N - kx)/2^{n-m}}{x}\right)$$

Putting it all together: asymptotic costs

Main result: Circuit for factoring n-bit integers $N = p^2 q$, with $q < 2^m$

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Depth: $\mathcal{O}(n)$
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Fast mult. + fast GCD:

Gates: $\mathcal{O}(n \log m)$
Depth: $\widetilde{\mathcal{O}}(n/m + m)$
Space: $\widetilde{\mathcal{O}}(m)$

Aside: fast multiplication in low space

[GDKM, Yao; arXiv:2403.18006]

New quantum multiplication circuit:

- Gates: $\mathcal{O}_\epsilon(t^{1+\epsilon})$
- Ancillas: zero!

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Parallel version of that circuit:

- **Depth:** $\mathcal{O}_\epsilon(t^\epsilon)$
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Parallel version of that circuit:

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<u>This mult. + standard GCD:</u>

Gates: $\mathcal{O}_\epsilon(nm^\epsilon + m^2)$

Depth: $\mathcal{O}_\epsilon((n/m)^{1+\epsilon} + m)$

Space: $\mathcal{O}(m)$

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Main result: Circuit for factoring n-bit integers $N = p^2 q$, with $q < 2^m$

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Gates: $\mathcal{O}(nm)$
Depth: $\mathcal{O}(n + m)$
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Fast mult. + fast GCD:

Gates: $\mathcal{O}(n \log m)$
Depth: $\widetilde{\mathcal{O}}(n/m + m)$
Space: $\widetilde{\mathcal{O}}(m)$

What should we set m to?

What integers should we apply it to?

Classical factoring: for integers $N = p^2 q$, with $n = \log N$ and $m = \log q$

General Number Field Sieve:

Used for RSA integers

Costs roughly $\exp\left(\mathcal{O}(\sqrt[3]{n})\right)$

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Set $m = \mathcal{O}(n^{2/3})$ for the *cheapest* quantum circuit classically as hard as RSA

Putting it all together: asymptotic costs

Main result: Circuit for factoring n-bit integers $N = p^2q$, with $\log q = m = O(n^{2/3})$

Schoolbook mult. + standard GCD:

Gates: $\mathcal{O}(n^{5/3})$
Depth: $\mathcal{O}(n)$
Space: $\mathcal{O}(n^{2/3})$

Fast mult. + fast GCD:

Gates: $\widetilde{\mathcal{O}}(n)$
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Space: $\widetilde{\mathcal{O}}(n^{2/3})$

Space $2m + o(m)$ seems achievable.
Classically-hard factoring with a few hundred qubits?

Summary and open questions

Factoring certain n-bit integers $N = p^2 q$ in:

- **Gates:** $\widetilde{\mathcal{O}}(n)$
- **Space** and **depth:** $\widetilde{\mathcal{O}}(n^{2/3})$

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- Optimization of concrete circuits
- Can this be generalized?
  - Currently: completely factor any integer with **distinct exponents** in prime factorization
  - Further generalizations? RSA??

Questions?

Greg
Kahanamoku-Meyer

Seyoon
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