Recent advances in quantum factoring

Greg Kahanamoku-Meyer

May 26, 2025

> Can the reader say what two numbers multiplied together will produce the number 8,616,460,799?
>
> I think it unlikely that anyone but myself will ever know.
>
> -William Stanley Jevons, 1874

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2032: Device with 50,000 qubits constructed

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Future B

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I want to live in Future A!

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Other less important reasons:

- Factoring makes a really straightforward efficiently-verifiable proof of quantumness
- The math is really fun

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2019 Greg

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With some set number of physical qubits below EC threshold, can double code distance if we use algorithm B---exponential decrease in logical error rate!

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Hot take: Right now, we should only really care about logical qubit count.

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We should optimize for depth if:

We have very good devices and we care about wall time.

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We should optimize for depth if:

We have very good devices and we care about wall time.

We should optimize for gate count if:

We have pretty good devices that are limited by magic production.

All (poly-time) quantum factoring algorithms: period finding

Find some $f_N(x)$ w/ period T, where T can be used to find factors

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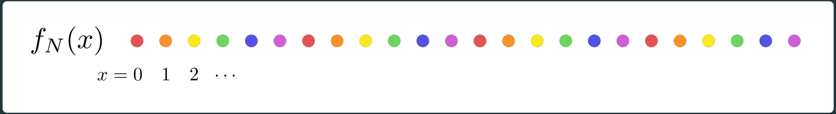
Find some $f_N(x)$ w/ period T, where T can be used to find factors

$\rightarrow$

Generate

$$\sum_{x=0}^{\mathrm{poly}(T)} |x\rangle\, |f_N(x)\rangle$$



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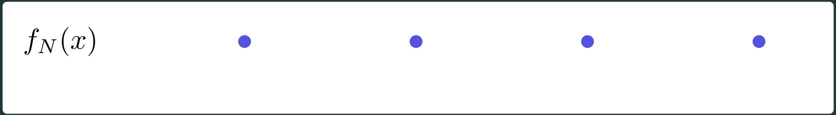
Find some $f_N(x)$ w/ period T, where T can be used to find factors → Generate $\sum_{x=0}^{\mathrm{poly}(T)} |x\rangle\,|f_N(x)\rangle$ → Measure $f_N(x)$, yielding superposition over a coset



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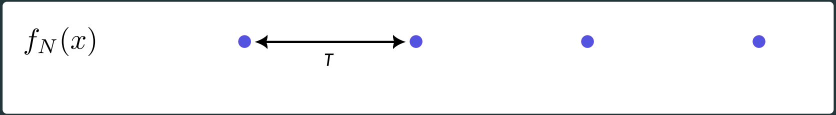

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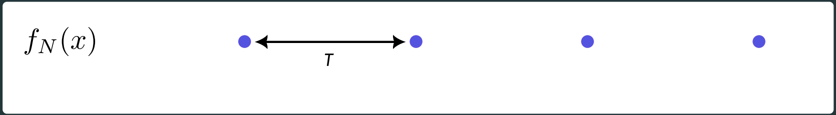

Qubit cost

- Input $|x\rangle$
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- Workspace

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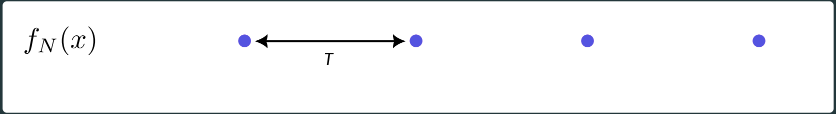

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Gate/depth cost

- Cost of $|f_N(x)\rangle$

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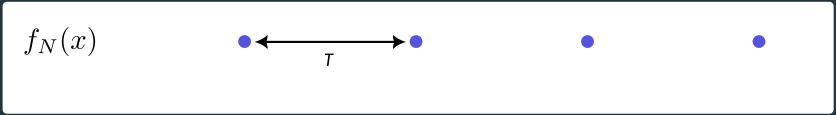

Qubit cost

- Input $|x\rangle$: $O(\log T)$ qubits
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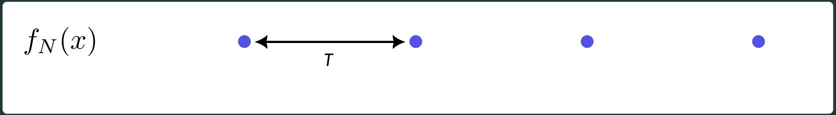

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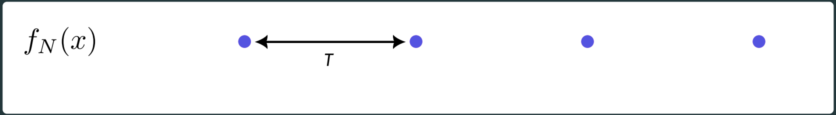

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Shor’s algorithm

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Function: $f_N(x) = a^x \bmod N$ **Period:** $T = \operatorname{ord}_N(a) \sim \mathcal{O}(N)$

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Function: $f_N(x) = a^x \bmod N$

Period: $T = \mathrm{ord}_N(a) \sim \mathcal{O}(N)$

Let $n = \lceil \log N \rceil$:

Qubit cost

- Input $|x\rangle$: $2n$ qubits
- Output $|f_N(x)\rangle$: n qubits
- Workspace: ???

Gate/depth cost

- Cost of $|f_N(x)\rangle$: ???

Reducing input to 1 (reused) qubit

Key observation (Zalka '98, and others):

$$a^x \bmod N = \prod_i c_i^{x_i} \bmod N$$

where $c_i = a^{2^i} \bmod N$.

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- Input $|x\rangle$: **1 qubit** (reused)
- Output $|f_N(x)\rangle$: n qubits
- Workspace: mult. workspace

Gate/depth cost

- Cost of $|f_N(x)\rangle$:
  - $2n$ multiplications

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[Gidney + Ekera '19]: Factoring 2048-bit $N = pq$ in 8 hours with 20 million physical qubits

Reducing output to 1 qubit: what if $f_N(x) : \{0,1\}^* \rightarrow \{0,1\}$?

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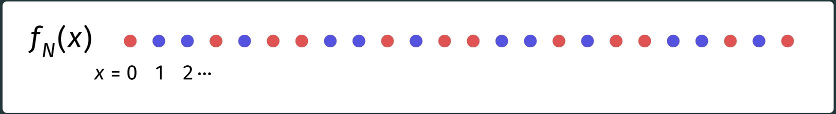

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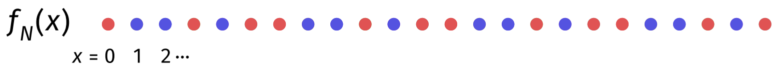

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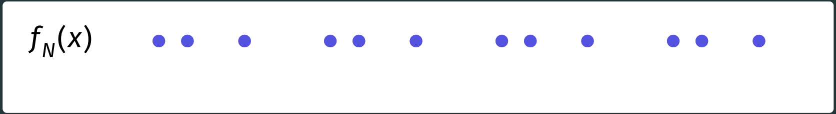

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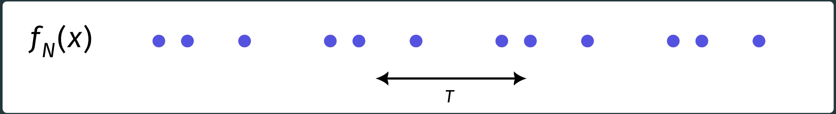

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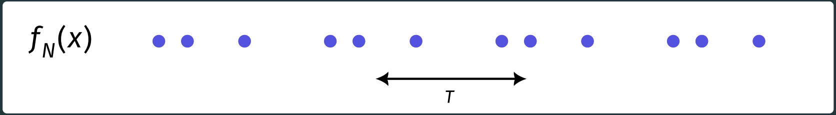

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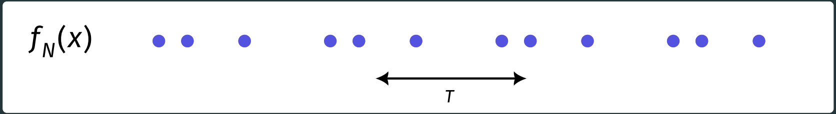

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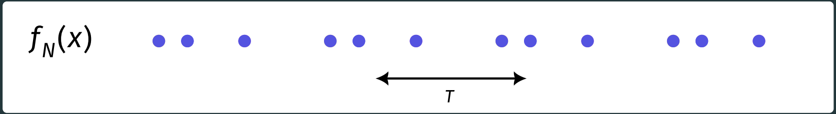

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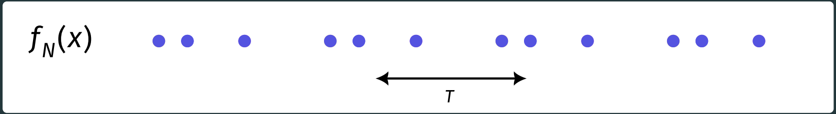

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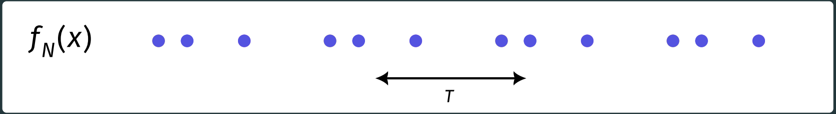
- Need $\sim \log T$ qubits for *input*
- Need workspace to compute $f_N(x)$

Avoid periods smaller than T



[May + Schlieper '22]: For some hash function $h : \{0,1\}^n \to \{0,1\}$, use

$$f_N(x) = h(a^x \bmod N)$$

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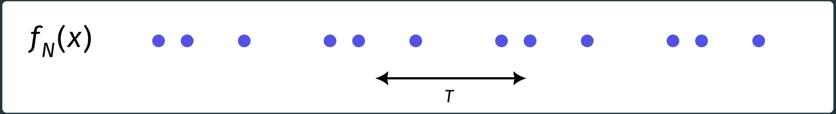

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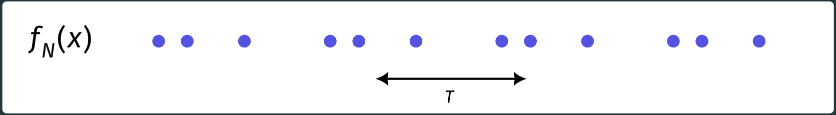
Qubit cost

- Input $|x\rangle$: 2n qubits
- Output $|f_N(x)\rangle$: **1 qubit**
- Workspace: $O(n)$ qubits

Gate/depth cost

- Cost of $|f_N(x)\rangle$:
  - $2n$ multiplications

Reduce period, reduce qubits for input



[Ekera + Hastad ’17]: Can factor $N = pq$ via discrete log with period $O(\sqrt{N})$

<u>Qubit cost</u>

- Input $|x\rangle$: **n/2 qubits**
- Output $|f_N(x)\rangle$: **1 qubit**
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<u>Gate/depth cost</u>

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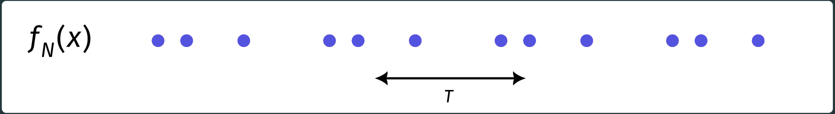
Reduce workspace



[Chevignard et al. '24]: Used residue number system to cut workspace to $\sim O(\log n)$ qubits

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- Output $|f_N(x)\rangle$: **1 qubit**
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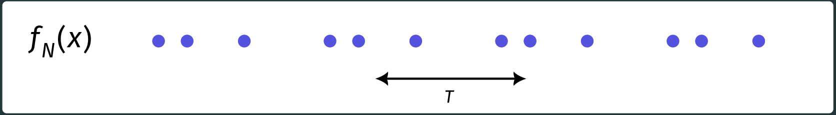
Gate/depth cost

- Cost of $|f_N(x)\rangle$:
  - $\sim$ 2 trillion Toffoli gates

Putting it all together



[Gidney last week]: Arithmetic + fault tolerance optimizations

Factor 2048-bit $N = pq$ using < 1 million physical qubits in ~ 1 week

Some things I've worked on

A sublinear space and depth factoring algorithm

For integers $N = P^2 Q$:
Gate count $\tilde{O}(n)$
Qubits and depth $\tilde{O}(n^{2/3})$

GDKM, S. Ragavan, V. Vaikuntanathan, K. Van Kirk. arXiv:2412.12558

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Log-depth "optimistic" QFT with no ancillas

Error bounded by ϵ on all but $O(\epsilon) \cdot 2^n$ basis states

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Shor's algorithm with:

$O(n^{2+\varepsilon})$ gates
$O(n^{1+\varepsilon})$ depth
$2n + O(n / \log n)$ total qubits

Factoring in sublinear space and depth

Greg Kahanamoku-Meyer

Seyoon Ragavan

Vinod Vaikuntanathan

Katherine Van Kirk

Asymptotic costs

Main result: Circuit for factoring n-bit integers $N = p^2 q$, with m-bit q

Schoolbook mult. + standard GCD:

Gates: $\mathcal{O}(nm)$
Depth: $\mathcal{O}(n + m)$
Space: $\mathcal{O}(m)$

Fast mult. + fast GCD:

Gates: $\widetilde{\mathcal{O}}(n)$
Depth: $\widetilde{\mathcal{O}}(n/m + m)$
Space: $\widetilde{\mathcal{O}}(m)$

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Classical factoring: for integers $N = p^2 q$, with $n = \log N$ and $m = \log q$

General Number Field Sieve:

Used for RSA integers

Costs roughly $\exp\left(\mathcal{O}(\sqrt[3]{n})\right)$

Lenstra ECM/Mulder:

Used for integers with small factors

Costs roughly $\exp\left(\mathcal{O}(\sqrt{m})\right)$

What should we set m to?

Classical factoring: for integers $N = p^2 q$, with $n = \log N$ and $m = \log q$

General Number Field Sieve:

Used for RSA integers

Costs roughly $\exp\left(\mathcal{O}(\sqrt[3]{n})\right)$

Lenstra ECM/Mulder:

Used for integers with small factors

Costs roughly $\exp\left(\mathcal{O}(\sqrt{m})\right)$

Set $m = \mathcal{O}(n^{2/3})$ for the *cheapest* quantum circuit classically as hard as RSA

Asymptotic costs

Main result: Circuit for factoring n-bit integers $N = p^2q$, with $\log q = m = O(n^{2/3})$

<u>Schoolbook mult. + standard GCD:</u>

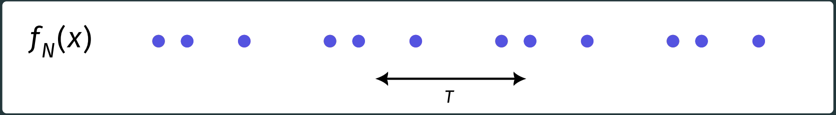
Gates: $\mathcal{O}(n^{5/3})$
Depth: $\mathcal{O}(n)$
Space: $\mathcal{O}(n^{2/3})$

<u>Fast mult. + fast GCD:</u>

Gates: $\widetilde{\mathcal{O}}(n)$
Depth: $\widetilde{\mathcal{O}}(n^{2/3})$
Space: $\widetilde{\mathcal{O}}(n^{2/3})$

Reducing output to 1 qubit: $f_N(x) : \{0,1\}^* \to \{0,1\}$

Find some $f_N(x)$ w/ period T, where T can be used to find factors → Generate $\sum_{x=0}^{\mathrm{poly}(T)} |x\rangle |f_N(x)\rangle$ → Measure $f_N(x)$, yielding superposition over a coset → QFT on x register, measure, postprocess to find T



Things we need for low qubit count:

- Small period T
- Avoid *smaller* periods than T
- Low workspace to compute $f_N(x)$

Some number theory

Legendre symbol

Legendre symbol

For a prime p:

$$\left(\frac{x}{p}\right) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod{p} \\ 1 & \text{if } \exists\, w \text{ s.t. } w^2 \equiv x \pmod{p} \\ -1 & \text{otherwise} \end{cases}$$

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Legendre symbol is 1) **efficient to compute** given x and p, 2) **periodic** with period p

Jacobi symbol

For a composite number $N = \prod_i p_i$:

$$\left(\frac{x}{N}\right) = \prod_i \left(\frac{x}{p_i}\right)$$

Jacobi symbol is 1) **efficient to compute** given x and N, 2) **periodic** with period...?

Some number theory

Jacobi symbol is 1) **efficient to compute** given x and N, 2) **periodic** with period...?

For $N = pq$:

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Period is N—not helpful for factoring!

Jacobi symbol is 1) **efficient to compute** given x and N, 2) **periodic** with period...?

For $N = p^2 q$:

$$\left(\frac{x}{N}\right) = \left(\frac{x}{p}\right)^2 \left(\frac{x}{q}\right)$$

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Period is q—exactly what we need!!

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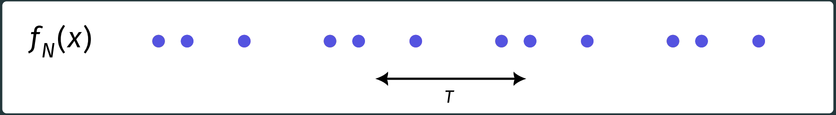

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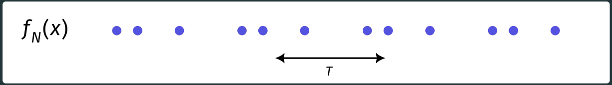

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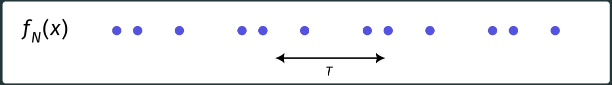

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Carl Friedrich Gauss, early 1800s: this function has the *ideal* Fourier spectrum for us!

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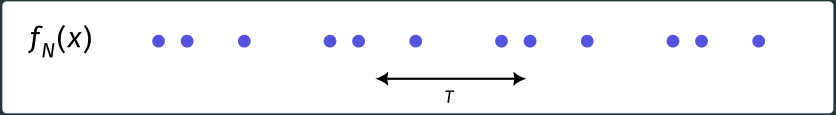

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People knew this!

"An Efficient Exact Quantum Algorithm for the Integer Square-free Decomposition Problem." Li, Peng, Du, Suter. Nature Scientific Reports, 2012.

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- Quantum period finding yields Q *exactly* if we take a superposition $x \in [0, N-1]$

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  - With superposition only to $\mathrm{poly}(Q)$, we still succeed $\to$ x needs only $\mathcal{O}(\log Q)$ qubits

Computing the Jacobi symbol

Goal: Compute $\left(\frac{x}{N}\right)$

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$$\left(\frac{a}{b}\right) \in \{-1, 0, 1\} \tag{1}$$

Computing the Jacobi symbol

Goal: Compute $|x\rangle \to \left(\frac{x}{N}\right)|x\rangle$

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Recall: N is classical, n bits; $|x\rangle$ is quantum, m qubits—and potentially $m \ll n$.

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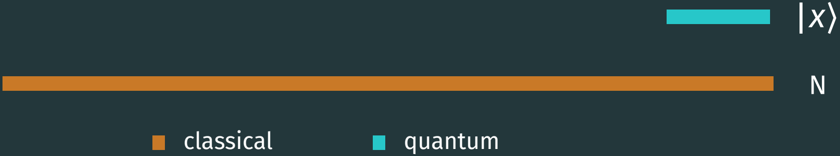
Yes!

Streaming classical bits to a quantum circuit

Key idea: can *stream* through the classical bits of N



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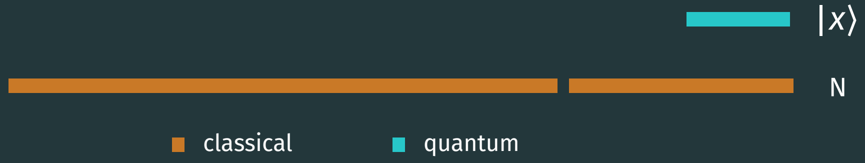

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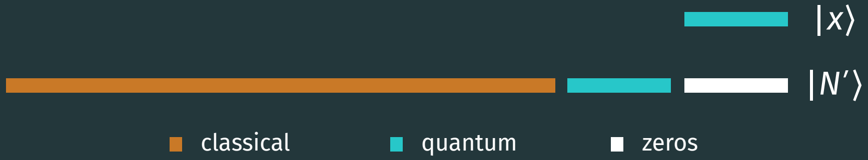

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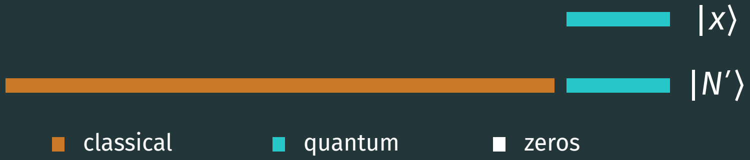

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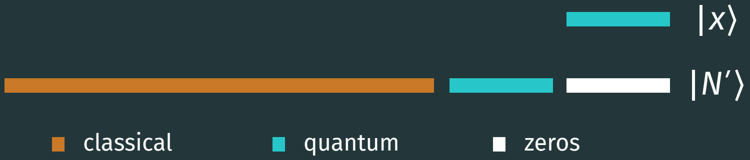

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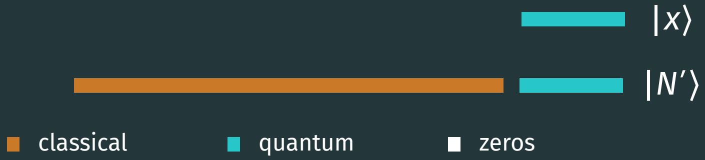

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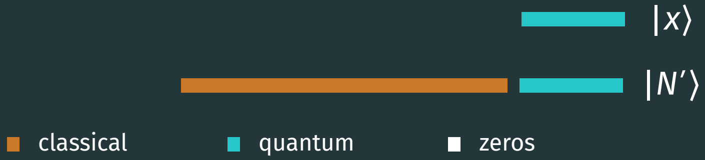

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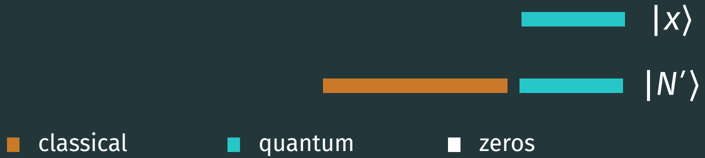

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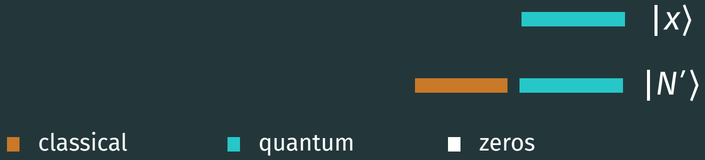

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classical quantum zeros

Streaming classical bits to a quantum circuit

Key idea: can *stream* through the classical bits of N

Now with two length-m inputs,
standard circuits for $\left(\frac{x}{N'}\right)$ have depth and qubits $\widetilde{O}(m)$

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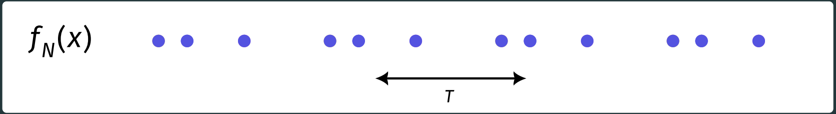

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Putting it all together: asymptotic costs

Main result: Circuit for factoring n-bit integers $N = p^2 q$, with $\log q = m = O(n^{2/3})$

<u>Schoolbook mult. + standard GCD:</u>

Gates: $\mathcal{O}(nm)$
Depth: $\mathcal{O}(n + m)$
Space: $\mathcal{O}(m)$

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Gates: $\widetilde{\mathcal{O}}(n)$
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Space $\sim 2m$ seems achievable, $m \sim 300$ seems classically hard.
Classically-hard factoring with a few hundred qubits?

Recent results

A sublinear space and depth factoring algorithm

For integers $N = P^2Q$:
Gate count $\tilde{O}(n)$
Qubits and depth $\tilde{O}(n^{2/3})$

GDKM, S. Ragavan, V. Vaikuntanathan, K. Van Kirk. arXiv:2412.12558

Log-depth "optimistic" QFT with no ancillas

Error bounded by ϵ on all but $O(\epsilon) \cdot 2^n$ basis states

GDKM, J. Blue, T. Bergamaschi, C. Gidney, I. Chuang. arXiv:2505.00701

Fast quantum integer multiplication

$O(n^{1+\varepsilon})$ gates
No ancilla qubits

GDKM, N. Yao. arXiv:2403.18006

Shor's algorithm with:

$O(n^{2+\varepsilon})$ gates
$O(n^{1+\varepsilon})$ depth
$2n + O(n/\log n)$ total qubits

Multiplication on quantum computers

Goal: Implement $\mathcal{U}_{c\times q}(a)\,|x\rangle\,|0\rangle = |x\rangle\,|ax\rangle$, for n-bit a and x

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$$\mathrm{QFT}\,|ax\rangle = \sum_z \exp\left(\frac{2\pi i a x z}{2^n}\right)|z\rangle$$

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[GDKM, Yao '24]: Can apply phase using:

$O(n^{1+\epsilon})$ gates $\quad O(n^{\epsilon})$ depth $\quad O(n/\log n)$ ancillas

A log-depth "optimistic" QFT with no ancillas

Greg
Kahanamoku-Meyer

John
Blue

Thiago
Bergamaschi

Craig
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Ike
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Structure of the QFT

The quantum Fourier transform on n qubits (dropping normalization):

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Let's do a similar trick, in base $b = 2^m$

QFT, block version

In base $b = 2^m$ we have $x = \sum_i 2^{mi} X_i$.

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ϵ-approximate QFT: since $m \sim O(\log(n/\epsilon))$, truncate to $0.X_i X_{i+1}$

Approximate QFT, block version

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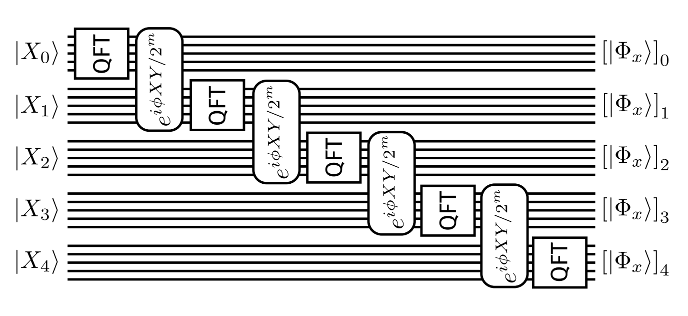
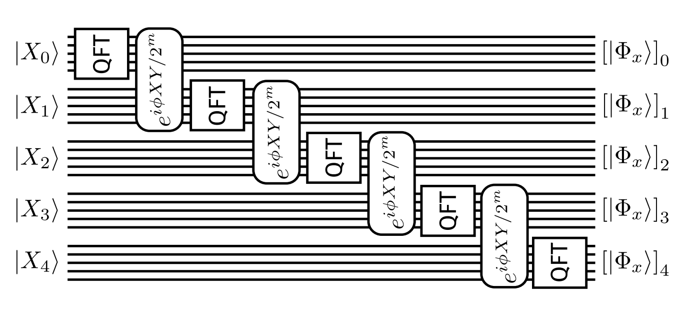
Approximate QFT, block version

In base $b = 2^m$ we have $x = \sum_i 2^{mi} X_i$. With $\phi = 2\pi/2^m$:

$$\mathrm{QFT}\,|x\rangle \equiv |\Phi_x\rangle \approx \bigotimes_{i=0}^{n/m-1} \left(\sum_{Y_j=0}^{2^m-1} e^{i\phi(X_i + X_{i+1}/2^m)Y_j} \left|Y_j\right\rangle \right)$$



Approximate QFT, block version



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Space-time product: $\mathcal{O}(n^2)$

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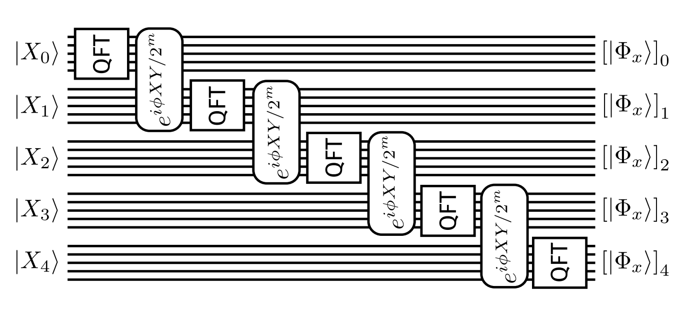

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Why are we stuck with linear depth here?

How to do better than linear depth

What happens if you apply $\mathrm{QFT}^\dagger$ to the following (remember $\phi = 2\pi/2^m$)

$$\mathrm{QFT}^\dagger \sum_{Y_j} e^{i\phi X_i Y_j} \left|Y_j\right\rangle = ?$$

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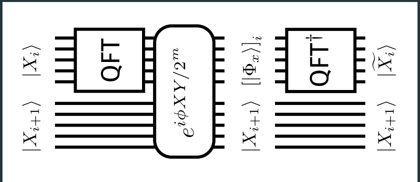
How to do better than linear depth

What happens if you apply $\mathrm{QFT}^\dagger$ to the following (remember $\phi = 2\pi/2^m$)

$$\mathrm{QFT}^\dagger \left[|\Phi_x\rangle\right]_i = \mathrm{QFT}^\dagger \sum_{Y_j} e^{i\phi(X_i + X_{i+1}/2^m)Y_j} \left|Y_j\right\rangle = ?$$



How to do better than linear depth

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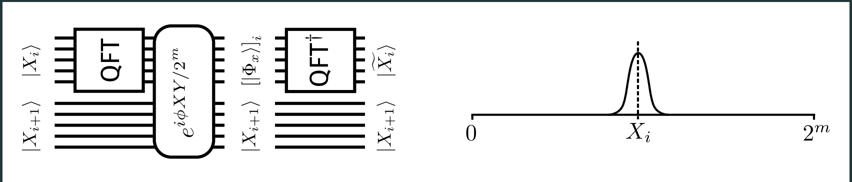
$$\mathrm{QFT}^\dagger\left[|\Phi_x\rangle\right]_i = \mathrm{QFT}^\dagger \sum_{Y_j} e^{i\phi(X_i + X_{i+1}/2^m)Y_j}\left|Y_j\right\rangle = \left|\widetilde{X}_i\right\rangle$$



A subtlety

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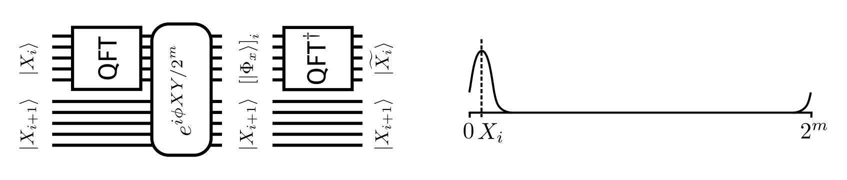
What happens if X_i is too close to $0 \pmod{2^m}$?



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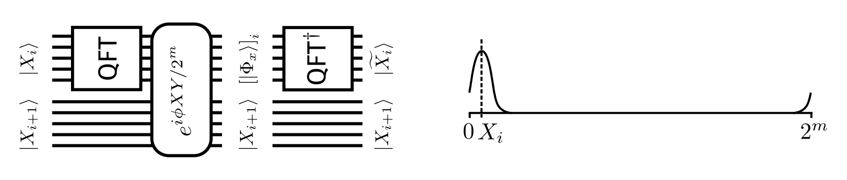
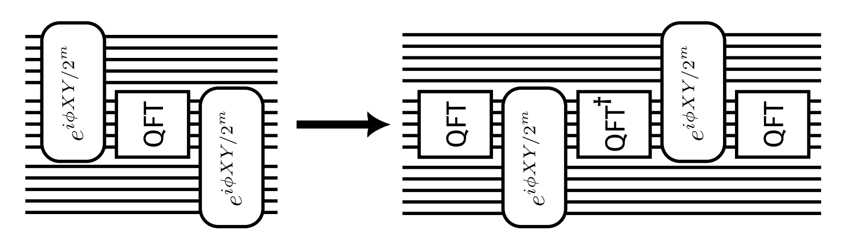


Part of the phase rotation "controlled off" $\left|\widetilde{X}_i\right\rangle$ will be off by 2^m!

Proposed replacement:



Rearranging gates

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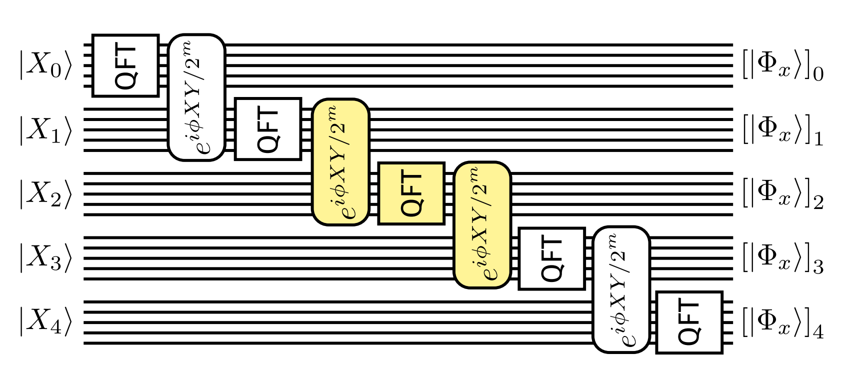

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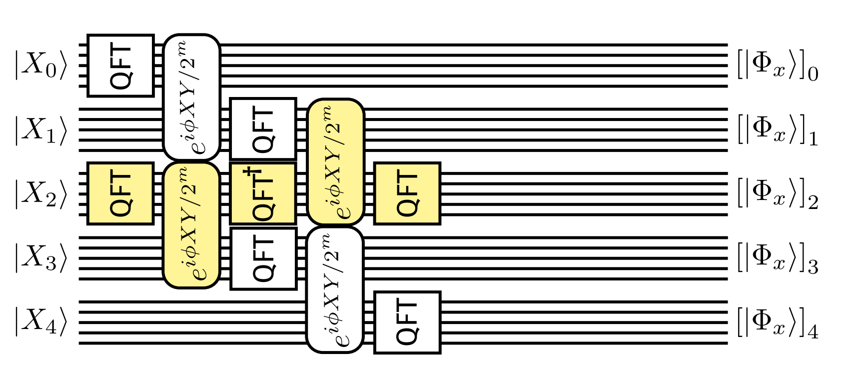
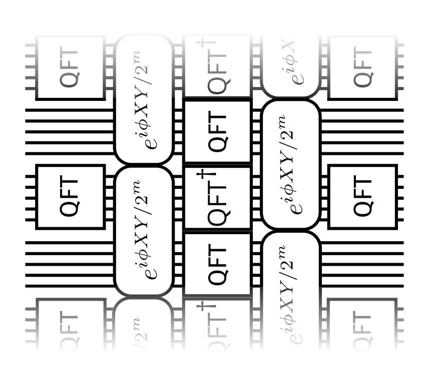
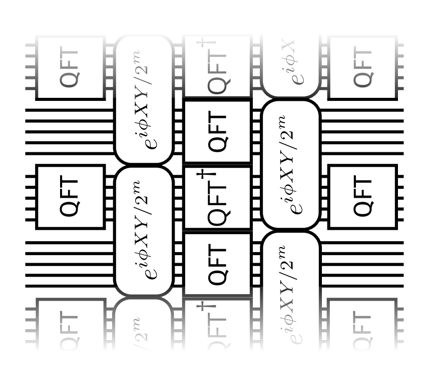


Looking a bit closer





Features:

- 5 layers, each layer has depth $\mathcal{O}(\log n)$

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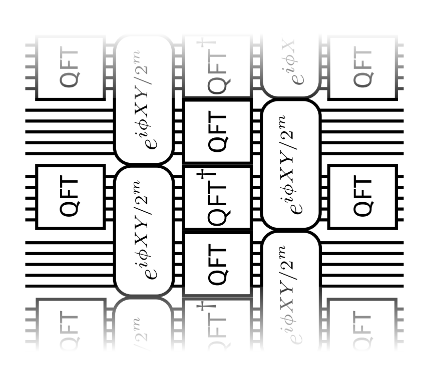

Features:

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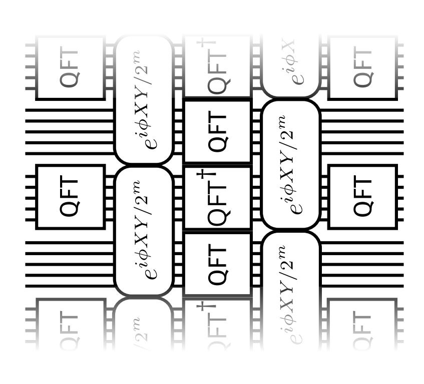
- No ancilla qubits



Features:

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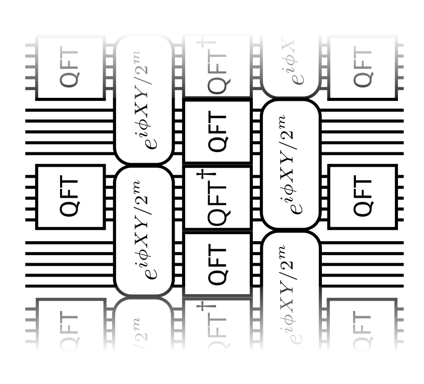
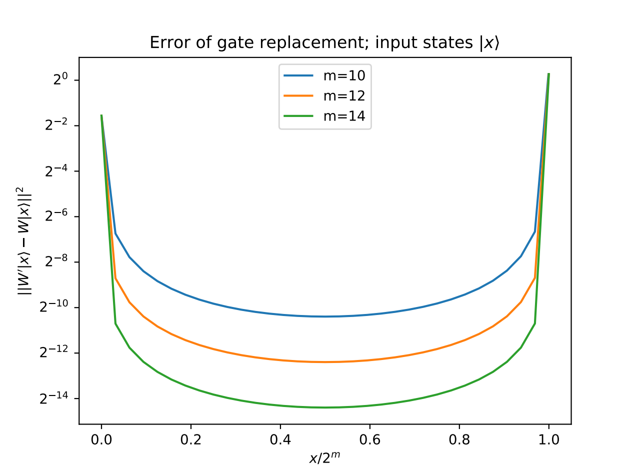


Features:

- 5 layers, each layer has depth $\mathcal{O}(\log n)$
- No ancilla qubits
- All gates have range at most $\mathcal{O}(\log n)$
- Doesn't give the right answer (sometimes)

“Optimistic” QFT

We have a good approximation on most basis states, with super nice properties!



“Optimistic” QFT

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“Optimistic” QFT

We have a good approximation on the vast majority of basis states, with super nice properties!

What should we do with it?

- Use it anyway (on “random” inputs)
- Bootstrap it into a slightly more expensive circuit that approximates QFT well on all basis states

Some things I've worked on

A sublinear space and depth factoring algorithm

For integers $N = P^2 Q$:
Gate count $\tilde{O}(n)$
Qubits and depth $\tilde{O}(n^{2/3})$

GDKM, S. Ragavan, V. Vaikuntanathan, K. Van Kirk. arXiv:2412.12558

Log-depth "optimistic" QFT with no ancillas

Error bounded by ϵ on all but $O(\epsilon) \cdot 2^n$ basis states

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Fast quantum integer multiplication

$O(n^{1+\varepsilon})$ gates
No ancilla qubits

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Shor's algorithm with:

$O(n^{2+\varepsilon})$ gates
$O(n^{1+\varepsilon})$ depth
$2n + O(n/\log n)$ total qubits

Fast quantum multiplication without ancillas

Greg
Kahanamoku-Meyer

Norm
Yao

Background: fast multiplication

Given two n-bit numbers x and y, write them as ”two digit” numbers in base $b = 2^{n/2}$.

$$
\begin{array}{rccc}
 & & x_1 & x_0 \\
\times & & y_1 & y_0 \\
\hline
 & & & x_0 y_0 \\
 & & x_1 y_0 & \\
 & & x_0 y_1 & \\
+ & x_1 y_1 & & \\
\hline
\end{array}
$$

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$$xy = x_1 y_1 b^2 + x_0 y_1 b + x_1 y_0 b + x_0 y_0$$

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\hline
\end{array}$$

$$xy = x_1 y_1 b^2 + x_0 y_1 b + x_1 y_0 b + x_0 y_0$$

Time remains $\mathcal{O}(n^2)$, because $4(n/2)^2 = n^2$

Background: Karatsuba multiplication

$$xy = x_1 y_1 b^2 + (x_0 y_1 + x_1 y_0) b + x_0 y_0$$

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$$xy = x_1 y_1 b^2 + (x_0 y_1 + x_1 y_0) b + x_0 y_0$$

Observation: $x_0 y_1 + x_1 y_0 = (x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0$

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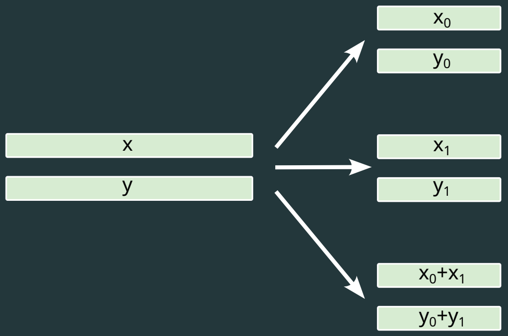
Can compute xy with only three multiplications of size $\log b = n/2$:

1. $x_1 y_1$
2. $x_0 y_0$
3. $(x_1 + x_0)(y_1 + y_0)$

Background: Karatsuba multiplication



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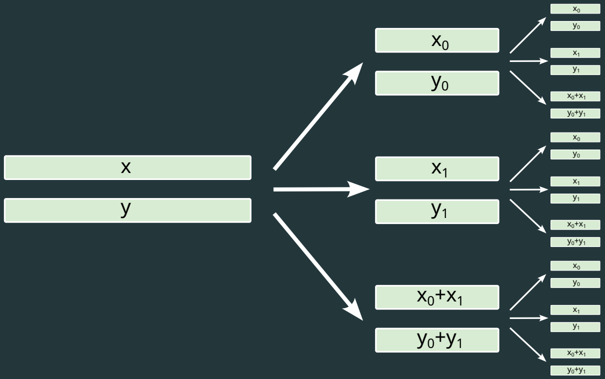

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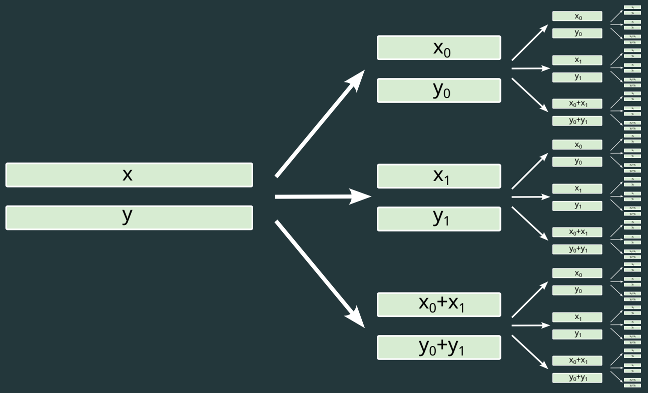

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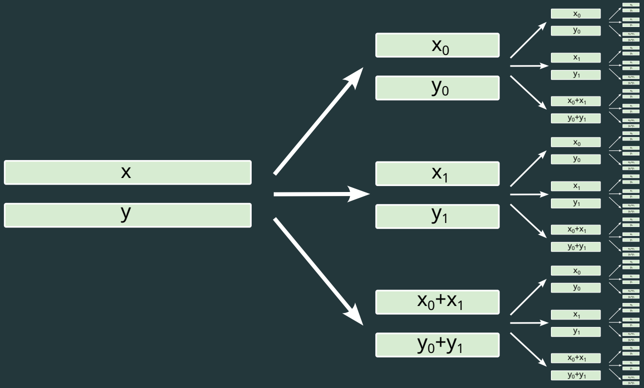

Depth: $d = \log_2 n$

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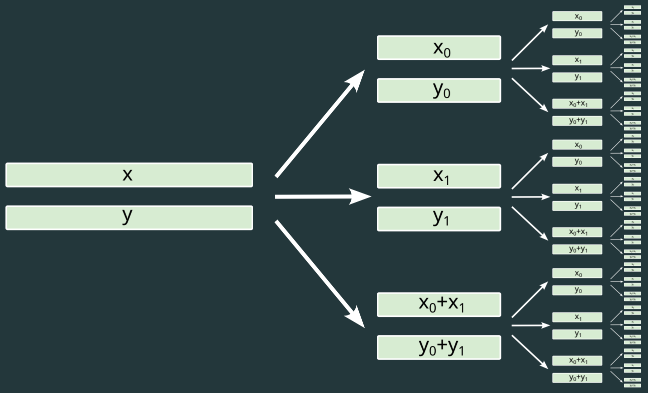

Depth: $d = \log_2 n$

Operations: 3^d

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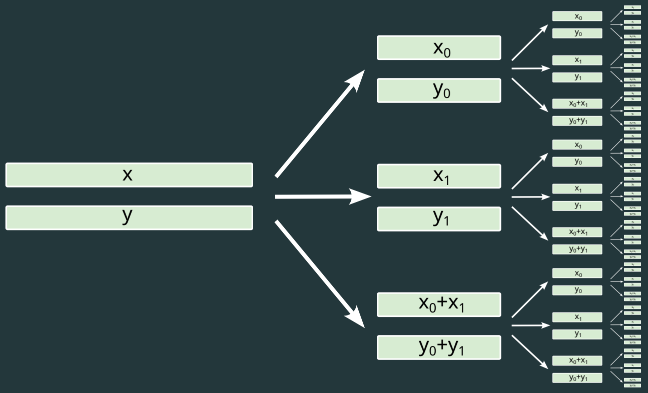

Depth: $d = \log_2 n$

Operations: 3^d

Cost: $\mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58\cdots})$

Fast classical-quantum multiplication

Goal: $\mathcal{U}(a)\,|x\rangle\,|0\rangle = |x\rangle\,|ax\rangle$

Fast classical-quantum multiplication

Goal: Apply phase ϕxz; x and z are quantum

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Karatsuba:

$$xz = 2^n x_1 z_1 + 2^{n/2}((x_0 + x_1)(z_0 + z_1) - x_0 z_0 - x_1 z_1) + x_0 z_0$$

Fast classical-quantum multiplication

Goal: Apply phase ϕxz; x and z are quantum

Plugging in Karatsuba:

$$\begin{aligned}\exp(i\phi xz) &= \exp(i\phi 2^n x_1 z_1)\\ &\cdot \exp(i\phi x_0 z_0)\\ &\cdot \exp\left(i\phi 2^{n/2}((x_0+x_1)(z_0+z_1) - x_0 z_0 - x_1 z_1)\right)\end{aligned}$$

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How are we supposed to reuse values in the *phase*?

Fast classical-quantum multiplication

Goal: Implement PhaseProduct$(\phi) |x\rangle |z\rangle = \exp(i\phi xz) |x\rangle |z\rangle$

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Goal: Implement $\text{PhaseProduct}(\phi)\,|x\rangle\,|z\rangle = \exp(i\phi xz)\,|x\rangle\,|z\rangle$

Re-ordering Karatsuba:

$$xz = (2^n - 2^{n/2})x_1 z_1 + 2^{n/2}(x_0 + x_1)(z_0 + z_1) + (1 - 2^{n/2})x_0 z_0$$

Fast classical-quantum multiplication

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Plugging in reordered Karatsuba:

$$\begin{aligned}\exp(i\phi xz) = \exp&\left(i\phi(2^n - 2^{n/2})x_1 z_1\right)\\ &\cdot \exp\left(i\phi(1 - 2^{n/2})x_0 z_0\right)\\ &\cdot \exp\left(i\phi 2^{n/2}(x_0 + x_1)(z_0 + z_1)\right)\end{aligned}$$

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Plugging in reordered Karatsuba:

$$
\begin{aligned}
\exp(i\phi xz) = &\exp(i\phi_1 x_1 z_1) \\
&\cdot \exp(i\phi_2 x_0 z_0) \\
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\end{aligned}
$$

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\phi_3 &= 2^{n/2}\phi
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Each of these has the same structure, but on half as many qubits $\rightarrow$ do it recursively!

Fast classical-quantum multiplication

Goal: Implement PhaseProduct(ϕ) $|x\rangle\,|z\rangle = \exp(i\phi xz)\,|x\rangle\,|z\rangle$

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Recursion relation: $T(n) = 3T(n/2)$

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Recursion relation: $T(n) = 3T(n/2) \Rightarrow \mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58\cdots})$ gates!

How many qubits do we need?

Splitting registers $|x\rangle \to |x_1\rangle |x_0\rangle$ and $|z\rangle \to |z_1\rangle |z_0\rangle$, can immediately do

- $\exp(i\phi_1 x_1 z_1)$
- $\exp(i\phi_2 x_0 z_0)$

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What about $\exp(i\phi_3 (x_0 + x_1)(z_0 + z_1))$?

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But, **addition is reversible** $\rightarrow$ do it *in-place*! E.g. $|x_1\rangle |x_0\rangle \rightarrow |x_1\rangle |x_0 + x_1\rangle$

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With a few tricks, can use zero ancillas.

Karatsuba

Multiply n-bit numbers via
3 multiplications of size $n/2$

$\mathcal{O}(n^{\log_2 3})$ gates

Making it go faster

Karatsuba

Multiply n-bit numbers via
3 multiplications of size $n/2$

$\mathcal{O}(n^{\log_2 3})$ gates

Toom-Cook

Multiply n-bit numbers via
$2k-1$ multiplications of size n/k

$\mathcal{O}(n^{\log_k (2k-1)})$ gates

Complexity vs. k

Toom-Cook has asymptotic complexity $\mathcal{O}(n^{log_k(2k-1)})$

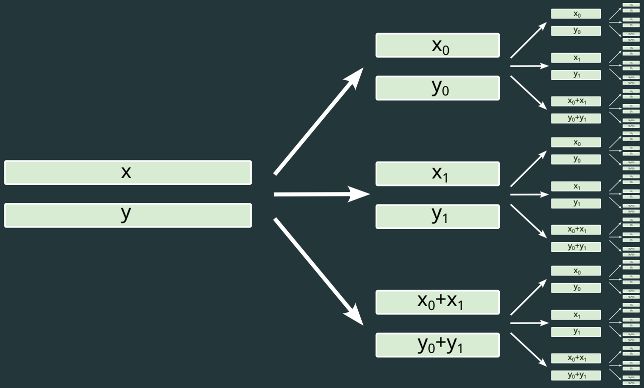
Complexity vs. k

Toom-Cook has asymptotic complexity $\mathcal{O}(n^{log_k(2k-1)})$

Algorithm	Gate count
Schoolbook	$\mathcal{O}(n^2)$
$k = 2$	$\mathcal{O}(n^{1.58\cdots})$
$k = 3$	$\mathcal{O}(n^{1.46\cdots})$
$k = 4$	$\mathcal{O}(n^{1.40\cdots})$
$\vdots$	$\vdots$

Depth

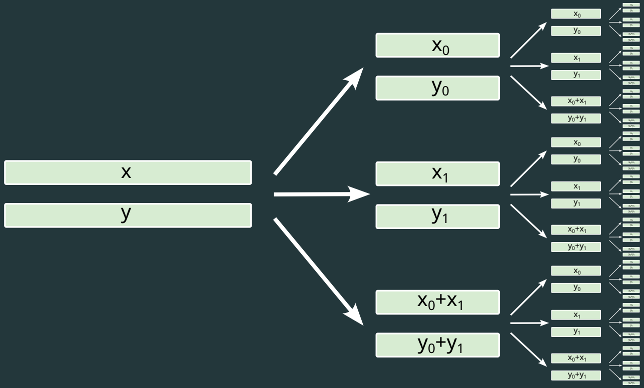
Parallelization is natural.



Depth

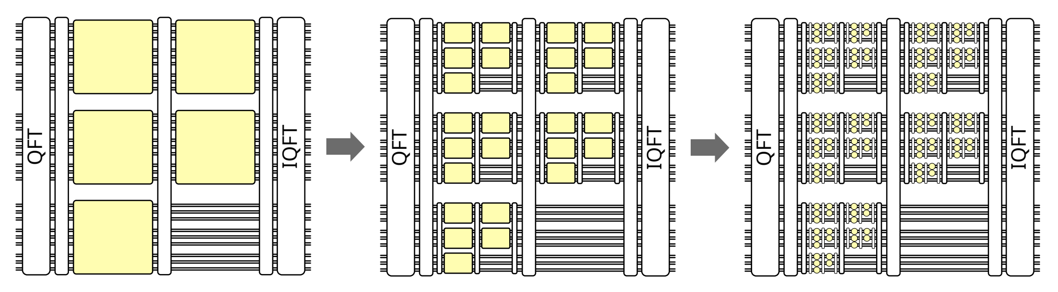
Parallelization is natural.

We have k sub-registers to work with—can do k sub-products in parallel.



Depth

$k = 3$:



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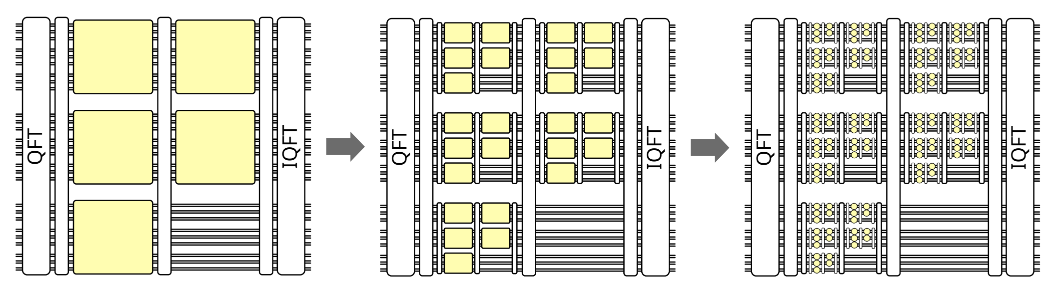

Depth is $\mathcal{O}(n^\epsilon)$ where $\epsilon = \log_k 2$, using $\mathcal{O}(n/\log n)$ ancillas.

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