Parallel Spooky Pebbling Makes Regev Factoring More Practical

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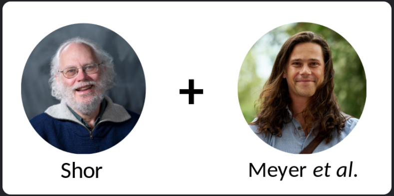
Circuits for factoring general-form numbers

Core of Shor's algorithm

To factor an n-bit number N, biggest cost is modular exponentiation in superposition:

$$f(a, x, N) = a^x \bmod N$$

Circuits for factoring general-form numbers



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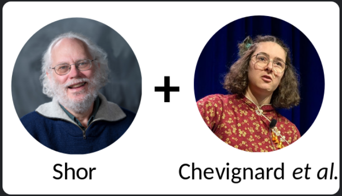

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$$f(a, x, N) = a^x \text{ mod } N$$

Gates: $\widetilde{O}(n^2)$ (per shot)

Core of Regev's algorithm

$$f(\vec{a}, \vec{x}, N) = \prod_i a_i^{x_i} \text{ mod } N$$

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Qubits: $\widetilde{O}(n)$

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Qubits: $O(n^{3/2})$

Algorithms for exponentiation

Intuition: how to compute $a^x \bmod N$ when x is a power of two?

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$$a \to a^2 \to a^4 \to a^8$$

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General case: multiply in extra factors of a along the way...

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Issue: squaring mod N is not reversible!

General problem: How to do $|x\rangle|0\rangle \to |x\rangle|f(x)\rangle$ efficiently when f has many irreversible steps?

Aside: avoiding irreversibility

What if we restructure our computation so each step is reversible?

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To compute $a^x \bmod N$

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Replace **squaring** with **multiplication by constants**, which is reversible!

To compute $a^x \bmod N = \prod_i \left(a^{2^i}\right)^{x_i} \bmod N$:

- **classically** precompute $a^2, a^4, a^8, \cdots \pmod{N}$
- Multiply together for all i where bit $x_i = 1$

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Regev

Regev's factoring speedup comes from multiplied values being **small**

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😢 $a_j^{2^i}$ mod N in general *not* small

- Shor's rearrangement would **kill improvement** in gate count

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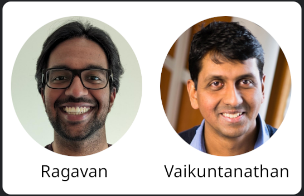

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but with **Fibonacci number**
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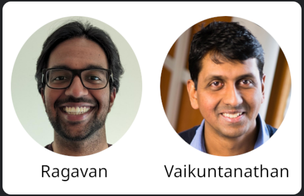

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- Reduces qubit count to $O(n)$

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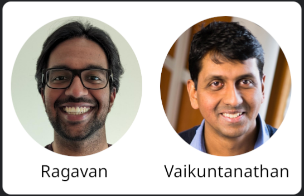

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😢 Large constant factors → Shor still wins in practice

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Let's see what we can do if we **accept irreversibility** of each step

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Consider an algorithm f with k steps f_i.

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Is it possible to do better?

Pebble games for reversible computation

Bennett ’89 introduced pebble games:

- Placing a pebble → computing a value (uses new space)

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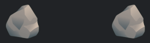

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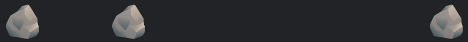

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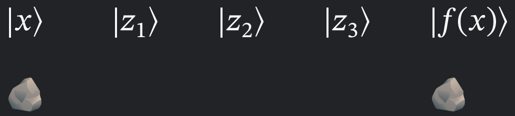

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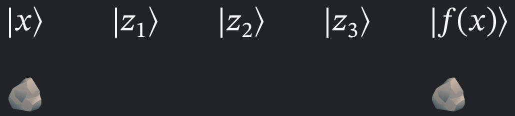

Space usage (max # pebbles): $O(k)$ registers
Time cost (# of steps): $2k - 1$ steps (optimal)

Using less space: a recursive strategy

1. Pebble from start to $k/2$

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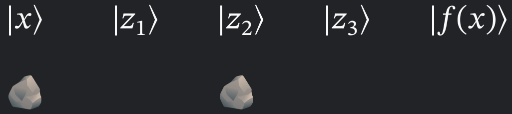

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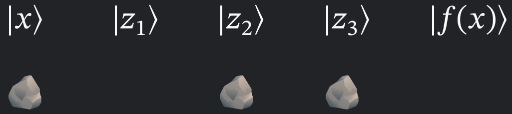

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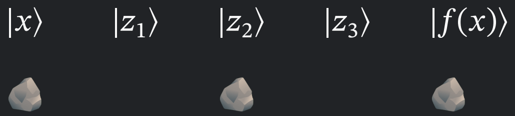

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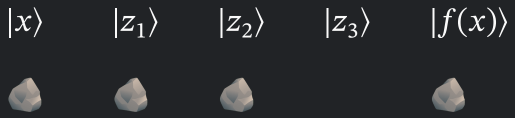

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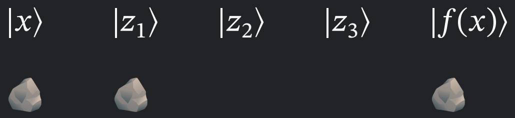

Using less space: a recursive strategy

1. Pebble from start to $k/2$
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3. Remove pebble at $k/2$

$$|x\rangle \quad |z_1\rangle \quad |z_2\rangle \quad |z_3\rangle \quad |f(x)\rangle$$

Space usage (max # pebbles): $O(\log k)$ registers 😃

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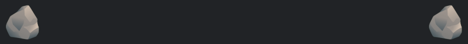

Space usage (max # pebbles): $O(\log k)$ registers 😃

Time cost (# of steps): $T(k) = 3T(k/2)$

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Space usage (max # pebbles): $O(\log k)$ registers 😃

Time cost (# of steps): $O(k^{\log_2 3}) \approx O(k^{1.58\cdots})$ steps 😢

Pebbling, but make it quantum

"Spooky Pebble Games and Irreversible Uncomputation"
algassert.com/post/1905

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Measurement-based uncomputation

Given an intermediate value

$$|z_i\rangle$$

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what if we apply $H^{\otimes n}$ and then measure it?

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$$(-1)^{d \cdot z_i}$$

where d is classically-known measurement outcome.

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Measurement-based uncomputation

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what if we apply $H^{\otimes n}$ and then measure it? Get phase

$$(-1)^{d \cdot z_i}$$

where d is classically-known measurement outcome.

We've turned $|z_i\rangle$ into a ghost!

Spooky pebble games

Rules:

- can only place or remove a pebble if there is a pebble to its left

Spooky pebble games

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- can only place or remove a pebble if there is a pebble to its left
- can ghost a pebble at any time

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- can ghost a pebble at any time—but must exorcise later by placing a pebble again

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$|x\rangle$

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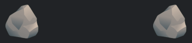
$|x\rangle$ $|z_1\rangle$



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$|x\rangle$ $\quad (-1)^{d_1 \cdot z_1}$ $\quad |z_2\rangle$

🪨 👻 🪨

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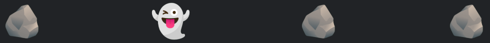
$$|x\rangle \qquad (-1)^{d_1 \cdot z_1} \qquad |z_2\rangle \qquad |z_3\rangle$$



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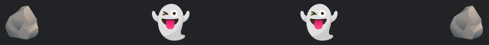

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- can ghost a pebble at any time—but must exorcise later by placing a pebble again

$\lvert x\rangle$	$(-1)^{d_1 \cdot z_1}$	$(-1)^{d_2 \cdot z_2}$	$\lvert z_3\rangle$	$\lvert f(x)\rangle$
🪨	👻	👻	🪨	🪨

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🪨	👻	👻	👻	🪨

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🪨 👻 👻 👻 🪨

We've been tempted too strongly by the power of dark magic...

Spooky pebble games

1. Blast straight to $k/2$, leaving ghosts

$|x\rangle$

Spooky pebble games

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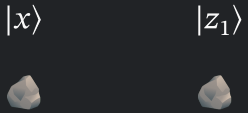

Spooky pebble games

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Spooky pebble games

1. Blast straight to $k/2$, leaving ghosts
2. Pebble from $k/2$ to k

$|x\rangle$ $(-1)^{d_1 \cdot z_1}$ $|z_2\rangle$

🪨 👻 🪨

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$|x\rangle$ $\quad$ $(-1)^{d_1 \cdot z_1}$ $\quad$ $|z_2\rangle$ $\quad$ $|z_3\rangle$

🪨 👻 🪨 🪨

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$\|x\rangle$	$(-1)^{d_1 \cdot z_1}$	$\|z_2\rangle$	$\|z_3\rangle$	$\|f(x)\rangle$
🪨	👻	🪨	🪨	🪨

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🪨 👻 🪨 🪨 🪨

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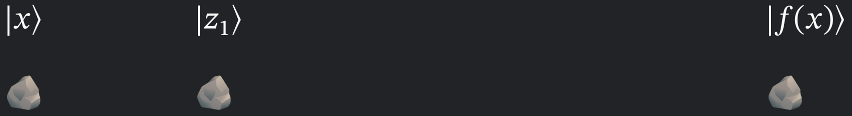

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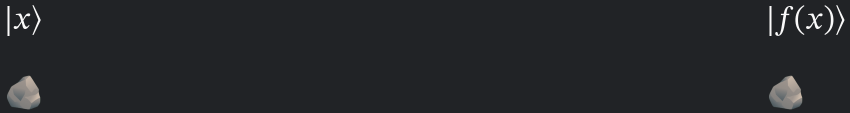

Space: $O(\log k)$ pebbles 😃

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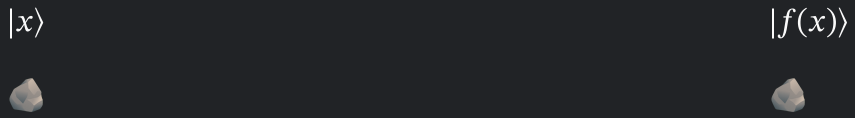

Space: $O(\log k)$ pebbles 😃

Time cost (# steps): $T(k) = O(k) + 2T(k/2)$

Spooky pebble games

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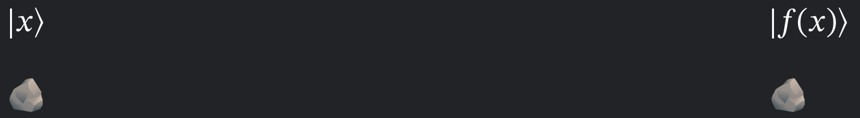

Space: $O(\log k)$ pebbles 😃

Time cost (# steps): $O(k \log k)$ steps 🤩

Our work: parallel spooky pebble games

Absolutely optimal depth for a length-k pebble game: $2k - 1$ steps

Our work: parallel spooky pebble games

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Without parallelism, this is **only** achieved by trivial $O(k)$-space strategy

Our work: parallel spooky pebble games

Absolutely optimal depth for a length-k pebble game: $2k - 1$ steps

Without parallelism, this is **only** achieved by trivial $O(k)$-space strategy

Can we achieve depth $2k - 1$ with less space, using parallelism?

An optimal parallel spooky pebble game for $k = 12$

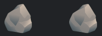
$|x\rangle \quad |z_1\rangle \quad |z_2\rangle \quad |z_3\rangle \quad |z_4\rangle \quad |z_5\rangle \quad |z_6\rangle \quad |z_7\rangle \quad |z_8\rangle \quad |z_9\rangle \quad |z_{10}\rangle \quad |z_{11}\rangle \quad |f(x)\rangle$

Step number: 0

Max. pebble count: 1

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle \quad |z_1\rangle \quad |z_2\rangle \quad |z_3\rangle \quad |z_4\rangle \quad |z_5\rangle \quad |z_6\rangle \quad |z_7\rangle \quad |z_8\rangle \quad |z_9\rangle \quad |z_{10}\rangle \quad |z_{11}\rangle \quad |f(x)\rangle$



Step number: 1

Max. pebble count: 2

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

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Step number: 2

Max. pebble count: 2

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

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Step number: 3

Max. pebble count: 2

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

Step number: 4

Max. pebble count: 2

An optimal parallel spooky pebble game for $k = 12$

$\|x\rangle$	$\|z_1\rangle$	$\|z_2\rangle$	$\|z_3\rangle$	$\|z_4\rangle$	$\|z_5\rangle$	$\|z_6\rangle$	$\|z_7\rangle$	$\|z_8\rangle$	$\|z_9\rangle$	$\|z_{10}\rangle$	$\|z_{11}\rangle$	$\|f(x)\rangle$
🪨	👻	👻	👻	🪨	🪨							

Step number: 5

Max. pebble count: 3

An optimal parallel spooky pebble game for $k = 12$

$\|x\rangle$	$\|z_1\rangle$	$\|z_2\rangle$	$\|z_3\rangle$	$\|z_4\rangle$	$\|z_5\rangle$	$\|z_6\rangle$	$\|z_7\rangle$	$\|z_8\rangle$	$\|z_9\rangle$	$\|z_{10}\rangle$	$\|z_{11}\rangle$	$\|f(x)\rangle$
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Step number: 6

Max. pebble count: 3

An optimal parallel spooky pebble game for $k = 12$

Step number: 7

Max. pebble count: 3

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle \quad |z_1\rangle \quad |z_2\rangle \quad |z_3\rangle \quad |z_4\rangle \quad |z_5\rangle \quad |z_6\rangle \quad |z_7\rangle \quad |z_8\rangle \quad |z_9\rangle \quad |z_{10}\rangle \quad |z_{11}\rangle \quad |f(x)\rangle$

Step number: 8

Max. pebble count: 4

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

Step number: 9

Max. pebble count: 4

An optimal parallel spooky pebble game for $k = 12$

$\|x\rangle$	$\|z_1\rangle$	$\|z_2\rangle$	$\|z_3\rangle$	$\|z_4\rangle$	$\|z_5\rangle$	$\|z_6\rangle$	$\|z_7\rangle$	$\|z_8\rangle$	$\|z_9\rangle$	$\|z_{10}\rangle$	$\|z_{11}\rangle$	$\|f(x)\rangle$
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Step number: 10

Max. pebble count: 5

An optimal parallel spooky pebble game for $k = 12$

$\|x\rangle$	$\|z_1\rangle$	$\|z_2\rangle$	$\|z_3\rangle$	$\|z_4\rangle$	$\|z_5\rangle$	$\|z_6\rangle$	$\|z_7\rangle$	$\|z_8\rangle$	$\|z_9\rangle$	$\|z_{10}\rangle$	$\|z_{11}\rangle$	$\|f(x)\rangle$
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Step number: 11

Max. pebble count: 6

An optimal parallel spooky pebble game for $k = 12$

$$|x\rangle \quad |z_1\rangle \quad |z_2\rangle \quad |z_3\rangle \quad |z_4\rangle \quad |z_5\rangle \quad |z_6\rangle \quad |z_7\rangle \quad |z_8\rangle \quad |z_9\rangle \quad |z_{10}\rangle \quad |z_{11}\rangle \quad |f(x)\rangle$$

Step number: 12

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

$$|x\rangle \quad |z_1\rangle \quad |z_2\rangle \quad |z_3\rangle \quad |z_4\rangle \quad |z_5\rangle \quad |z_6\rangle \quad |z_7\rangle \quad |z_8\rangle \quad |z_9\rangle \quad |z_{10}\rangle \quad |z_{11}\rangle \quad |f(x)\rangle$$

Step number: 13

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

$\lvert x\rangle$	$\lvert z_1\rangle$	$\lvert z_2\rangle$	$\lvert z_3\rangle$	$\lvert z_4\rangle$	$\lvert z_5\rangle$	$\lvert z_6\rangle$	$\lvert z_7\rangle$	$\lvert z_8\rangle$	$\lvert z_9\rangle$	$\lvert z_{10}\rangle$	$\lvert z_{11}\rangle$	$\lvert f(x)\rangle$
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Step number: 14

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

$\vert x\rangle$	$\vert z_1\rangle$	$\vert z_2\rangle$	$\vert z_3\rangle$	$\vert z_4\rangle$	$\vert z_5\rangle$	$\vert z_6\rangle$	$\vert z_7\rangle$	$\vert z_8\rangle$	$\vert z_9\rangle$	$\vert z_{10}\rangle$	$\vert z_{11}\rangle$	$\vert f(x)\rangle$
🪨	👻	👻	👻	🪨	🪨	👻	🪨	🪨				🪨

Step number: 15

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

Step number: 16

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

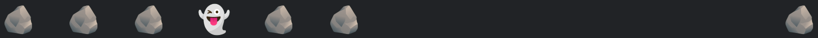
$|x\rangle \quad |z_1\rangle \quad |z_2\rangle \quad |z_3\rangle \quad |z_4\rangle \quad |z_5\rangle \quad |z_6\rangle \quad |z_7\rangle \quad |z_8\rangle \quad |z_9\rangle \quad |z_{10}\rangle \quad |z_{11}\rangle \quad |f(x)\rangle$

Step number: 17

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

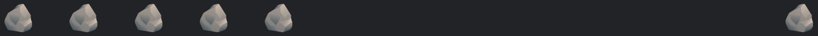


Step number: 18

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

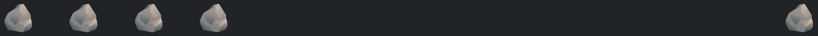


Step number: 19

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

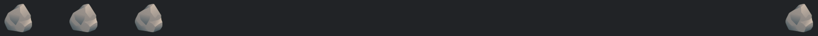


Step number: 20

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

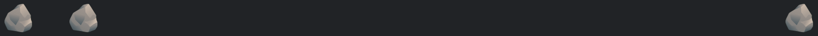


Step number: 21

Max. pebble count: 7

An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$



Step number: 22

Max. pebble count: 7

$|x\rangle \quad |z_1\rangle \quad |z_2\rangle \quad |z_3\rangle \quad |z_4\rangle \quad |z_5\rangle \quad |z_6\rangle \quad |z_7\rangle \quad |z_8\rangle \quad |z_9\rangle \quad |z_{10}\rangle \quad |z_{11}\rangle \quad |f(x)\rangle$

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An optimal parallel spooky pebble game for $k = 12$

$|x\rangle$ $|z_1\rangle$ $|z_2\rangle$ $|z_3\rangle$ $|z_4\rangle$ $|z_5\rangle$ $|z_6\rangle$ $|z_7\rangle$ $|z_8\rangle$ $|z_9\rangle$ $|z_{10}\rangle$ $|z_{11}\rangle$ $|f(x)\rangle$

🪨 🪨

Step number: $23 = 2k - 1$, which is optimal 🤩

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Our results

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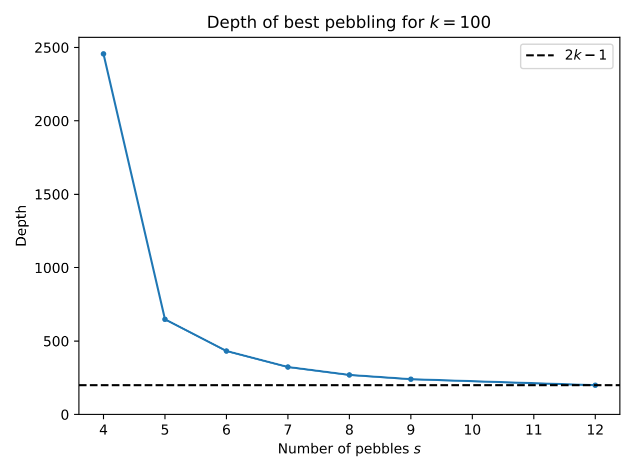
- Finds lowest-depth solution for *any* fixed number of pebbles s and length k

Numerical results



Factoring results

For factoring 4096-bit RSA:

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Note: this is shamelessly focusing on **depth** our best metric...

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- "This paper creates **slack**"
- Efficiently evaluating sequential algorithms is **widely applicable**
- **Generalization:** parallel spooky pebbling on arbitrary graphs

INTELLIGENCE COMMUNITY
POSTDOCTORAL RESEARCH
FELLOWSHIP PROGRAM

VE RI TAS

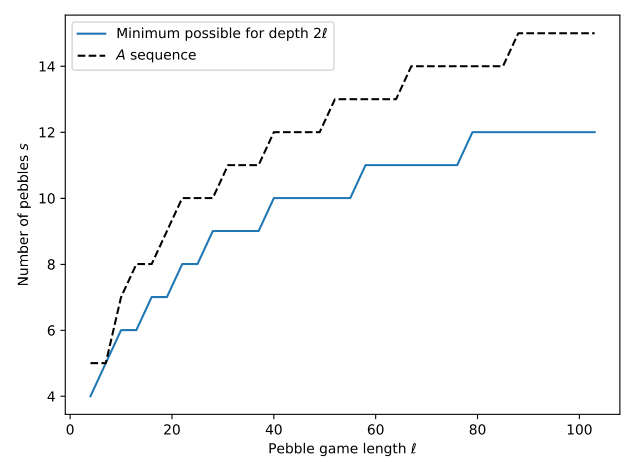
Hertz Foundation

Thank you!

arXiv:2510.08432

Backup

Numerical results



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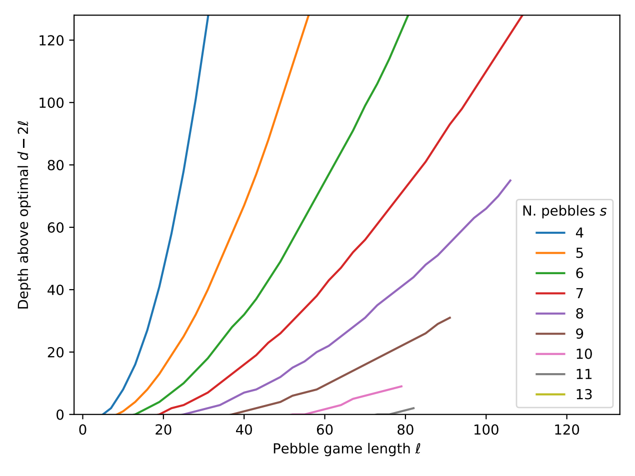

Making Shor reversible

Break up x into its individual bits x_i:

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where classical $c_i = a^{2^i} \bmod N$

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Each iteration is a controlled multiplication by classical c_i—which is reversible!